# APPLICATION OF THE LINEAR FLOW DIFFUSIVITY EQUATION IN ESTIMATING WATER INFLUX IN LINEAR WATER DRIVE RESERVOIRS

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## ABSTRACT

Over the years, many models for estimating unsteady-state water influx in edge–water drive reservoir-aquifer systems as well as bottom-water drive reservoir–aquifer systems have been developed. Unfortunately, little emphasis has been placed on reservoir–aquifer systems of linear geometry.

Therefore, this paper examines the applicability of the linear flow diffusivity equation in developing a model suitable for estimating unsteady-state water influx in linear water drive reservoir–aquifer (infinite) systems.

The linear flow diffusivity equation is written in dimensionless form by defining an appropriate dimensionless time and dimensionless length in order to enhance a more generalized application.

Moreover, the required constant-terminal pressure solution of the dimensionless equation is obtained by imposing the appropriate Drichlet's and Newmann's boundary conditions.

Finally, the applicability of the solution is demonstrated using superposition principle.

## 1.0 Introduction

When an oil reservoir and the adjoining aquifer are contained between two parallel and sealing faulting planes, the flow of fluid is essentially parallel to these planes and is "Linear". Furthermore, such reservoir is said to be producing under linear water drive.

Over the years, the likes of Van Everdingen and Hurst, Cater–Tracy and Fetkovich have developed models for estimating unsteady– state water influx in edge–water drive reservoir-aquifer systems. Coats on the other hand presented a model for bottom water drive reservoir-aquifer systems. Unfortunately, little emphasis has been placed on reservoir aquifer systems of linear geometry. This paper presents a water influx model suitable for estimating unsteady–state water influx in reservoir-aquifer (infinite) systems of linear geometry.

Finally, since water influx in an essential term of the material balance equation (MBE) for water drive reservoirs, therefore reservoir engineers would find the result of this study extremely useful in estimating water influx required for aquifer fitting.

## 2.0 The Governing Equation

The mathematical modeling of Fluids flow in porous media requires the combination of at least 3 equations, these are;

a. The Transport Equation
b. The Continuity Equation
c. The Equation of State

a. **The Transport Equation:** This is basically the Darcy's equation (for flow of slightly compressible fluids) or the Forcheimer's Equation (for flow of compressible fluids). Since the focus of this study is on linear flow, therefore we would be considering the forms of these equations applicable to linear flow Geometry.

The Darcy's equation for linear flow geometry is given by:

$$V = -\frac{k}{\mu}\frac{dp}{dx} \quad \text{-------------- (1)}$$

On the other hand, the linear form of the Forcheimer's equation is given by

$$-\frac{dp}{dx} = \frac{\mu v}{k} + F_t \rho v^2 \quad \text{-------------- (2)}$$

b. **The Continuity Equation**: This is essentially a conservation of mass equation. It is a mathematical expression of the relationship between the mass influx, mass out-flux and mass accumulation in porous media.

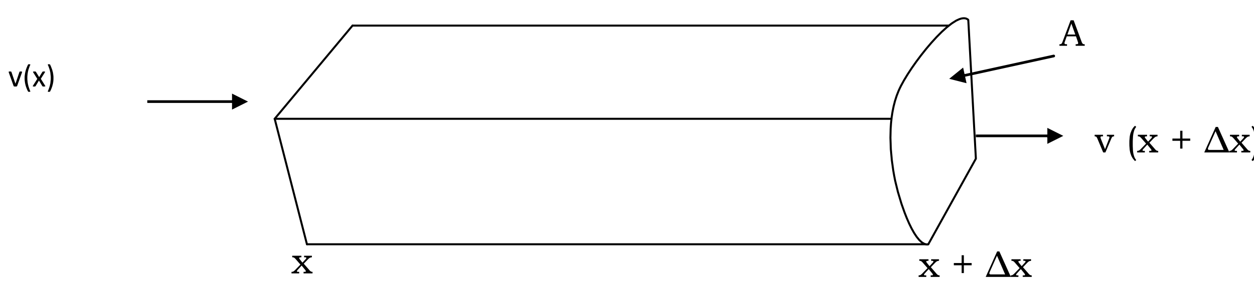


Fig. (1)

Applying the conservation of mass principle to the above core sample of Area A and length Δx, we have:

Mass influx - mass out-flux = mass accumulation

Mass influx into the sample = $A(x)\ \rho(x)\ v(x)\ \Delta t$ ----- (i)

Mass flux out of the sample = $A(x + \Delta x)\ \rho(x+\Delta x)\ v(x+\Delta x)\ \Delta t$ -----(ii)

Mass accumulation = $m(t+\Delta t) - m(t)$ --------- (iii)

Therefore, we have

$$(A(x)\ \rho(x)v(x) - A(x+\Delta x)\ \rho(x+\Delta x)\ v(x+\Delta x))\ \Delta t = m(t+\Delta t) - m(t)$$

For linear flow geometry, the Area A exposed to flow

is constant. Therefore, we have

$$-A\left[\rho v(x+\Delta x) - \rho v(x)\right] = \frac{m(t+\Delta t) - m(t)}{\Delta t} \text{ ------- (iv)}$$

as $\Delta t \to 0$, we have

$$-A\left[\rho v(x+\Delta x) - \rho v(x)\right] = \frac{dm}{dt}$$

But $m = \rho V = \rho\phi V_p = \rho\phi A\ \Delta x$ ------------(v)

Substituting (v) into (iv) we have

$$-\ A\left[\rho v(x+\Delta x) - \rho v(x)\right] = \frac{d(\rho\phi)A\Delta x}{dt}$$

Dividing both sides by $A\Delta x$ and letting $\Delta x \to 0$, Then we have

$$-\frac{d(\rho v)}{dx} = \frac{d(\rho\phi)}{dt} \text{-----------(3)}$$

Equation (3) is the continuity equation for one–dimensional linear flow in porous media.

c. **<u>The Equation of State:</u>** This is basically the compressibility equation. The conpressibility equation can be written in terms of volume, density or formation volume factor of the fluid. The compressibility equation is expressed mathematically as:

$$c = -\frac{1}{V}\frac{\partial V}{\partial p} \text{ (in terms of vloume) ---------(vi)}$$

$$c = -\frac{1}{\rho}\frac{\partial \rho}{\partial p} \quad \text{(in terms of density)} \quad \text{------- (vii)}$$

$$c = \frac{-1}{Bi}\frac{\partial Bi}{\partial p} \quad \text{(in terms of formation volume factor ) –(viii)}$$

Another compressibility equation of interest is that of the formation which can be expressed in terms of volume or porosity. The formation compressibility is expressed mathematically as:

$$c_f = \frac{-1}{V_f}\frac{\partial V_f}{\partial p} \quad \text{---------- (ix)}$$

$$c_f = \frac{1}{\phi}\frac{\partial \phi}{\partial p} \quad \text{-------- (x)}$$

The partial differntial equation (PDE) that governs linear flow of slightltly compressibile fluid(oil and water) in porous media can be derived by combining equation (1), (3), (vii) and (x) as follows:

$$V = \frac{-k}{\mu}\frac{dp}{dx} \quad \text{-----------------(1)}$$

$$-\frac{\partial(\rho v)}{\partial x} = \frac{\partial(\rho\phi)}{\partial t} \quad \text{--------------(3)}$$

By putting equation (1) into (3) and applying chain rule to the right hand side of equation (3) we have

$$\frac{\partial}{\partial x}\left(\frac{k\rho}{\mu}\frac{dp}{dx}\right) = \phi c\frac{\partial p}{\partial t}$$

Expanding the left hand side of the above equation, we have

$$\frac{\partial^2 p}{\partial x^2} = \frac{\phi\mu c}{k}\frac{\partial p}{\partial t} \quad \text{----------------------(4)}$$

Equation (4) is the diffusivty equation for linear flow of fluids in porous media and the rest of this study focuses on how equation (4) can be used to develop a water influx model suitable for estimating water influx in reservoir-aquifer system of linear geometry.

## 2.1 Non-dimensionalization

The presentation of equations in dimensionless form is of great importance in the fields of science and engineering as it enhances a more generalized application of results and solutions. In this study, the linear flow diffusivity equation is written in dimensionless form by defining appropriate dimesnionless time and length as follows:

$$t_D = \frac{t}{T}, \qquad x_D = \frac{x}{L}$$

Where:

$t_D$ = Dimensionless time

$x_D$ = Dimensionless length

$L$ = Characteristic length

$T$ = Characteristic time

To write equatin (4), in term of $t_D$ and $x_D$ we have:

$$\frac{\partial^2 p}{\partial x^2} = \frac{\phi\mu c}{k}\frac{\partial p}{\partial t} \text{ ------------------(4)}$$

Since:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_D} \cdot \frac{\partial x_D}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x_D}$$

$$\therefore \quad \frac{1}{L^2}\frac{\partial}{\partial x_D}\left(\frac{\partial p}{\partial x_D}\right) = \frac{\phi\mu c}{k}\frac{\partial p}{\partial t}$$

Also,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_D} \cdot \frac{\partial t_D}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t_D}$$

Therefore, we have:

$$\frac{1}{L^2}\frac{\partial^2 p}{\partial x^2_D} = \frac{\phi\mu c}{kT}\frac{\partial p}{\partial t_D}$$

$$\therefore \quad \frac{\partial^2 p}{\partial x^2_D} = \frac{\phi\mu c L^2}{kT}\frac{\partial p}{\partial t_D}$$

Since $T = \frac{t}{t_D}$ we have

$$\frac{\partial^2 p}{\partial x^2_D} = \frac{\phi\mu c L^2 t_D}{kt}\frac{\partial p}{\partial t_D} \qquad \text{----------------(5)}$$

Where $t_D = \frac{kt}{\phi\mu c L^2}$

Equation (5) is the dimensionless linear flow diffusivity equation.

## 2.2 Solution of the Dimensionless Equation:

The solution of the dimensionless diffusivity equation appropriate for estimating water influx in reservior-aqufier systems is the constant-terminal pressure solution(CTPS). The CTPS requires keeping the presure at the inner boundary constant while observing the rate of water influx from the aquifer into the reservoir.

The solution of equation(4) requires specifying the initial and boundary conditions.

### 2.2.1 Initial Condition (IC)

The IC would be presented both in terms of pressure and pressure drop respectively. This is due to the fact that the IC in terms of pressure drop would be required for obtaining the generalized solution for Linear flow in the later part of this study. At time zero the pressure at all points in the formation is constant and equal to unity and also the pressue drop is equal to zero. Therefore, we have

$P(x_{D},0) = 1$

$\Delta P\ (x_{D},0) = 0$

### 2.2.2 Inner Boundary Condition (IBC)

For the same reason as in (2.2.1) above, the inner boundary condtion would also be presented in terms of rate of influx (Newmann's condition) and pressure drop (Drichlet's condition). When the reservoir is opened, the pressure at the reservoir-aquifer boundary, $x_D = 0$, immediately drops to zero and remains zero for the duration of the production history.

Moreover, the rate of water influx at the reservoir–aquifer boundary is governed by Darcy's law.

Therefore we have:

$$\Delta p(0, t_D) = 1$$

$$q = \frac{kwh}{\mu L}\left(\frac{\partial p}{\partial x_D}\right)_{x_D = 0}$$

### 2.2.3 Outer Boundary Condition (OBC)

The outer boundary condition depends on the size of the aquifer i.e. whether finite (Bounded) or infinite.

#### Finite aquifers:

For finite aquifers, the pressure at the outer boundary changes and also it is assumed that there is no flow across the boundary.

$$\left(\frac{\partial p}{\partial x_D}\right)_{x_D = \beta} = 0$$

$\beta < 100$ (the aquifer is less than 100 times the size of the reservior).

#### Infinite aquifers:

For infinite aquifers, the pressure at the outer boundary is constant and equal to unity( i.e initial pressure) and also there is no pressure drop at the outer boundary.

$$P(x_D, t_D) = 1$$

$$x_D \rightarrow \infty$$

$$\Delta p(x_D, t_D) = 0$$

$$x_D \rightarrow \infty$$

## 2.2.4 Cumulative Water Influx

From the above IBC, we have

$$q = \frac{kwh}{\mu L}\left(\frac{\partial p}{\partial x_D}\right)_{x_D} = 0$$

The above equation can be written as

$$\frac{\partial Q}{\partial t} = \frac{kwh}{\mu L}\left(\frac{\partial p}{\partial x_D}\right)_{x_D = 0}$$

Writing the LHS of the above equation in dimensionless form we have:

$$\frac{1}{T}\frac{\partial Q}{\partial t_D} = \frac{kwh}{\mu L}\left(\frac{\partial p}{\partial x_D}\right)_{x_D = 0}$$

$$\frac{\partial Q}{\partial t_D} = \frac{kwhT}{\mu L}\left(\frac{\partial p}{\partial x_D}\right)_{x_D} = 0$$

Since $T = \frac{\phi\mu cL^2}{K}$

$$\frac{\partial Q}{\partial t_D} = \frac{kwh}{\mu L} \cdot \frac{\phi\mu cL^2}{k}\left(\frac{\partial p}{\partial x_D}\right)_{x_D = 0}$$

$$\frac{\partial Q}{\partial t_D} = \phi cLwh\left(\frac{\partial p}{\partial x_D}\right)_{x_D} = 0$$

Integrating, we have;

$$Q = \phi\, clwh \int^{t_D} \left(\frac{\partial p}{\partial x_D}\right) \partial t_D \qquad \text{-----(6)}$$

$\overline{\partial x_D}_{\;x_D=0}$

0

Equation (6) can be written as

$Q = \phi clwh\, Q_{tD}$

:. $Q = UQt_D$ -----------(7)

where

$$Qt_D = \int_0^{t_D} \left( \frac{\partial p}{\partial x_D} \right)_{x_D=0} \partial t_D$$

And $U = \phi clwh$

For any pressure drop $\Delta p$, equation (7) gives

$Q = U\Delta P\, Qt_D$ ------------ (8)

Equation (8) is the generalized expression for cumlative water influx in a reservoir–aquifer systems of linear geometry.

To estimate cumulative water influx using equation (8), an expression for the generalized solution for linear flow is required.

Therefore, the next phase of this study focuses on the method of obtaining the generalized solution for linear flow in order to be able to estimate cumulative water influx using equation (8).

## 2.2.5 The Genaralized Solution for Linear Flow

In order to apply equation (8), the generalized solution for linear flow ($Qt_D$) is required. This generalized solution can be obtained by solving the dimensionless linear flow diffusivity equation .

The dimensionless linear flow diffusivity equation (equation 5) as derived above is given by

$$\frac{\partial^2 p}{\partial x_D^2} = \frac{\partial p}{\partial t_D} \quad \text{----------------(5)}$$

Where $x_D = \frac{K}{L}$ and $t_D = \frac{kt}{\phi\mu c L^2}$

Interpreting pressure in equation (5) as pressure drop and presenting the IC and BC in terms of pressure drop then we have the following problem;

Governing Equation: $\frac{\partial^2 p}{\partial x^2_D} = \frac{\partial p}{\partial t_D}$ ------------------(5)

IC : $p(x_D, 0) = 0$

BC: Since the emphasis of this study is on infinite reservoir- aquifer systems, therefore we have

Inner BC: $p(0, t_D) = 1$

Outer BC: $p(x_D, t_D) = 0$

$x_D \rightarrow \infty$

The above problem would be solved by invoking the method of Laplace transform as follows :

Applying Lapace Transform to equation (5) and imposing the IC, we have:

$$\frac{\partial^2 p(x_D,s)}{\partial x^2_D} = sp(x_D,s) \text{ --------------------(9)}$$

Equation (9) is a second-order linear ordinary differential equation which can be solved analytically.

Assume a solution:

$P = e^{mx_D}$

$\frac{\partial^2 p}{\partial x^2_D} = m^2 e^{mx_D}$, therefore we have:

$m^2 e^{mx_D} - s e^{mx_D} = 0$

$e^{mx_D} \neq 0$

Therefore $m^2 - s = 0$

$m = \pm \sqrt{s}$

$P = A e^{(\sqrt{s})x_D} + B e^{-(\sqrt{s})x_D}$

Transforming the BC, we have

$P(0,s) = \frac{1}{s}$ (Inner BC)

$P(x_D,s) = 0$ (outer BC)

$X_D \rightarrow \infty$

Imposing the transformed outer BC, we have

$p(x_D, s) = A\, e^{(\sqrt{s})x_D} + Be^{-(\sqrt{s})x_D}$

$P(x_D, s) = Ae^{\infty\sqrt{s}} + Be^{-\infty\sqrt{s}} = 0$

$x_D \to \infty$

$Ae^{\infty\sqrt{s}} = 0$

$\therefore \quad A = 0$

$P(x_D, s) = Be^{-(\sqrt{s})x_D}$

Imposing the transformed inner BC, we have

$P(0,s) = Be^0 = \frac{1}{s}$

$\therefore B = \frac{1}{s}$

$\therefore P(x_D, s) = \frac{e^{-(\sqrt{s})x_D}}{s}$ -----------------------(10)

Equation (10) is the transformed form of the solution to equation (8) for an infinite reservoir-aquifer system.

From equation (10) we have

$$\frac{\partial p(x_D, s)}{\partial x_D} = \frac{-\sqrt{s}e^{-(\sqrt{s})x_D}}{s} = \frac{-e^{-(\sqrt{s})x_D}}{\sqrt{s}}$$

$$\left(\frac{\partial p(x_D, s)}{\partial x_D}\right)_{x_D = 0} = \frac{-1}{\sqrt{s}}$$

$$\left(-\frac{\partial p(x_D, s)}{\partial x_D}\right)_{x_D = 0} = \partial x_D \left(\frac{\partial p(x_D, s)}{}\right)_{x_D = 0}$$

$$\left(\frac{\partial p(x_D, s)}{\partial x_D}\right)_{x_D = 0} = \frac{1}{\sqrt{s}}$$

By the theorem of Laplace transform, we have

$$\int_0^{t_D} \left( \frac{\partial p(x_D, s)}{\partial x_D} \right)_{x_D = 0} \partial t_D = \frac{1}{s^{3/2}}$$

$$\int_0^{t_D} \left( \frac{\partial p(x_D, s)}{\partial x_D} \right)_{x_D = 0} \partial t_D = L \int_0^{t_D} \frac{\partial p(x_D, t_D)\ \partial t_D}{\partial x_D}$$

$$\int_0^{t_D} \left( \frac{\partial p(x_D, t_D)}{\partial x_D} \right)_{x_D = 0} \partial t_D = L^{-1} \int_0^{t_D} \left( \frac{\partial p(x_D, s)\ \partial t_D}{\partial x_D} \right)_{x_D = 0}$$

$$Qt_D = \int_0^{t_D} \left( \frac{\partial p(x_D, t_D)\ \partial t_D}{\partial x_D} \right)_{x_D = 0} = L^{-1} \left( \frac{1}{s^{3/2}} \right)$$

$$Qt_D = \frac{2\, t_D^{1/2}}{\sqrt{\pi}} \text{ -----------------------(11)}$$

Equation (11) is the generalized solution for linear flow. Putting equation (11) into equation (8) we have

$$Q = U\Delta P\, Qt_D = \frac{2\ U\Delta P\, t_D^{1/2}}{\sqrt{\pi}} \text{ ........... (12)}$$

Equation (12) can be used to estimate water influx at any time $t_n$.

## 3.0 Results and Discussion:

In practice the estimation of cumulative water influx involves the discretization of the continuous pressure decline curve into discrete pressure decline steps. Afterwards the cumulative water influx due to all the pressure decline steps is then estimated using superposition principle.

In the same vein, the expression for cumulative water influx obtained in this study must therefore be applied in practice using superposition principle.

Therefore, the cumulative water influx at the end of time, $t_n$ is given by:

$$Q = U\,(\Delta P_1 Qt_{Dn} + \Delta P_2 Qt_{Dn-1} + \Delta P_3\, Qt_{Dn-2} + ... + \Delta P_n Qt_{D1})$$

$$Q = U \sum_{i=1,\ j=n}^{i=n,\ j=1} \Delta P_i\, Qt_{Dj} \qquad \text{---------(13)}$$

Equation (13) can be used for practical estimation of water influx due to an infinite aquifer of linear geometry.

Where $\Delta P_i$ = pressure drop at the end of time $t_i$, i = 1,2, -----n

$$\Delta P_i = ½\,(P_{i-2} - P_i),\ i \geq 2 \qquad \text{-----(14)}$$

$Qt_{Dj}$ = Dimensionless water influx for time t =$t_j$, j= 1, 2, ----n

$$Qt_{Dj} = \frac{2\, t_{Dj}^{½}}{\sqrt{\pi}} \qquad \text{--------(15)}$$

$$t_Dj = \frac{\alpha k t_j}{\phi \mu c L^2} \qquad \text{-------(16)}$$

$\alpha$ = 0.000264 ($t_j$-hrs)

= 0.00634 ($t_j$ – days)

= 2.309 ($t_j$ – years)

U = 0.1781wL$\phi$hc (bbl/psi) ------(17)

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## Nomenclature

h = Reservoir thickness

K = Permeability of the medium of interest

P = Pressure

$\Delta$p = Pressure drop

Q = cumulativ water influx

$Qt_D$ = Generalized solution for linear flow.

U = Water influx constant

v = Velocity of flow

$V_p$ = Pore volume

V = Bulk volume

$V_f$ = Formation volume

W = Reservoir width

$\mu$ = Fluid's viscosity

ρ = Fluid density

$F_t$ = For chimer's coefficient

$\phi$ = Porosity

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