Approximation of Soft Fixed Points Using Multiplicative Analog of Zamfirescu Operators

Clement Ampadu
31 Carrolton Road
Boston, MA, 02132-6303, USA
drampadu@hotmail.com

Abstract: In this paper we introduce an iterative soft sequence to establish a convergence theorem to approximate soft fixed points of multiplicative Zamfirescu operators. The real counterpart of this iterative soft sequence gives the generalized Mann iteration scheme. We also introduce a new two step iterative soft scheme to approximate common soft fixed points for two asymptotically nonself mappings in multiplicative soft analog of Banach spaces. The real counterpart of this new two step iterative soft scheme gives the two step iteration scheme introduced by Thianwan [S. Thianwan, Common fixed points of the new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space, J.Comput Appl. Math. 224(2009), 688-695]

AMS Subject Classification: 47H10, 47H09

I. Introduction and Preliminaries

Definition I (Iterative Soft Scheme Under Consideration): Let $\tilde{X}$ be a multiplicative soft analog of normed linear space, and let $\tilde{K}$ be a nonempty, closed, multiplicative convex subset of $\tilde{X}$. Let $(T,\alpha),(S,\beta):\tilde{K}\to\tilde{K}$. Let $\{\widetilde{x_{n,\gamma}}\}$ with $\widetilde{x_{0,\gamma}}$ in $\tilde{K}$ be defined as $\widetilde{x_{n+1,\gamma}}=\widetilde{a_{n,\gamma}}\widetilde{x_{n,\gamma}}+\widetilde{b_{n,\gamma}}(T,\alpha)(\widetilde{x_{n,\gamma}})+\widetilde{c_{n,\gamma}}(S,\beta)(\widetilde{x_{n,\gamma}})$, $n=0,1,2,\ldots$, where $\{\widetilde{a_{n,\gamma}}\},\{\widetilde{b_{n,\gamma}}\},\{\widetilde{c_{n,\gamma}}\}$ are soft real sequences in $[\bar{0},\bar{1}]$ with $\widetilde{a_{n,\gamma}}+\widetilde{b_{n,\gamma}}+\widetilde{c_{n,\gamma}}=\bar{1}$ and $\widetilde{b_{n,\gamma}}+\widetilde{c_{n,\gamma}}=\widetilde{\delta_{n,\gamma}}$

Remark 2: (a) The real counterpart of the above iterative soft sequence is known as the generalized Mann iteration scheme in the literature

(b) When $(T,\alpha)=(S,\beta)$ in Definition 1, then the real counterpart of the iterative soft scheme obtained, gives the Mann iteration scheme in the literature

(c) If $(T,\alpha)=(S,\beta)$ in Definition 1, and $\widetilde{\delta_{n,\gamma}}=\widetilde{\delta_{\gamma}}$ (a soft constant), then the real counterpart of the iterative soft scheme obtained, gives the Krasnoselskij iteration scheme in the literature

(d) If $(T,\alpha)=(S,\beta)$ in Definition 1, and $\widetilde{\delta_{n,\gamma}}=\bar{1}$, then the real counterpart of the iterative soft scheme obtained, gives the Picard iteration scheme in the literature

Definition 3 (Iterative Soft Scheme Under Consideration): Let $\tilde{X}$ be a multiplicative soft analog of normed linear space, and let $\tilde{K}$ be a nonempty, closed, multiplicative convex subset of $\tilde{X}$. Let $(T,\alpha),(S,\beta):\tilde{K}\to\tilde{K}$ be two nonlinear operators. Let $\{\widetilde{x_{n,\gamma}}\}$ with $\widetilde{x_{0,\gamma}}$ in $\tilde{K}$ be defined as $\widetilde{x_{n+1,\gamma}}=\widetilde{a_{n,\gamma}}\widetilde{y_{n,\gamma}}+\widetilde{b_{n,\gamma}}(T,\alpha)(\widetilde{y_{n,\gamma}})+\widetilde{c_{n,\gamma}}(S,\beta)(\widetilde{y_{n,\gamma}})$, $\widetilde{y_{n,\gamma}}=(1-\widetilde{\zeta_{n,\gamma}})\,\widetilde{x_{n,\gamma}}+\widetilde{\zeta_{n,\gamma}}(T,\alpha)(\widetilde{x_{n,\gamma}})$

$n=0,1,2,\ldots$, where $\{\widetilde{a_{n,\gamma}}\},\{\widetilde{b_{n,\gamma}}\},\{\widetilde{c_{n,\gamma}}\}$, and $\{\widetilde{\zeta_{n,\gamma}}\}$ are soft real sequences in $[\bar{0},\bar{1}]$ with $\widetilde{a_{n,\gamma}}+\widetilde{b_{n,\gamma}}+\widetilde{c_{n,\gamma}}=\bar{1}$ and $\widetilde{b_{n,\gamma}}+\widetilde{c_{n,\gamma}}=\widetilde{\delta_{n,\gamma}}$.

Remark 4: (a) If $(T,\alpha)=(S,\beta)$ in Definition 3, then the real counterpart of the iterative soft scheme obtained gives the Thianwan two step iteration [see, S. Thianwan, Common fixed points of the new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space, J.Comput Appl. Math. 224(2009), 688-695]

(b) If $(T,\alpha)=(S,\beta)$ and $\widetilde{\zeta_{n,\gamma}}=\bar{0}$ in Definition 3, then the real counterpart of the iterative soft scheme obtained gives the Mann iteration scheme in the literature

(c) If $(T,\alpha)=(S,\beta)$, $\widetilde{\zeta_{n,\gamma}}=\bar{0}$, and $\widetilde{\delta_{n,\gamma}}=\widetilde{\delta_{\gamma}}$ (a soft constant) in Definition 3, then the real counterpart of the iterative soft scheme obtained, gives the Krasnoselskij iteration scheme in the literature

(d) If $(T,\alpha)=(S,\beta)$, $\widetilde{\zeta_{n,\gamma}}=\bar{0}$, and $\widetilde{\delta_{n,\gamma}}=\bar{1}$ in Definition 3, then the real counterpart of the iterative soft scheme obtained, gives the Picard iteration scheme in the literature

Definition 5: Let $(\tilde{X},\tilde{d})$ be a multiplicative soft metric space, and (T,α) be a self-map of $\tilde{X}$. We say (T,α) is a multiplicative Z-operator or a multiplicative Zamfirescu operator if it satisfies at least one of the following

(a) $\tilde{d}\left((T,\alpha)(\widetilde{x_\eta}),(T,\alpha)(\widetilde{y_\varrho})\right)\,\tilde{\leqq}\,\tilde{d}(\widetilde{x_\eta},\widetilde{y_\varrho})^{a}$, where $a\,\tilde{\in}\,[\bar{0},\bar{1})$ and $\widetilde{x_\eta},\widetilde{y_\varrho}$ is in $\tilde{X}$

(b) $\tilde{d}\left((T,\alpha)(\widetilde{x_\eta}),(T,\alpha)(\widetilde{y_\varrho})\right)\,\tilde{\leqq}\,\tilde{d}\left(\widetilde{x_\eta},(T,\alpha)(\widetilde{x_\eta})\right)^{b}.\tilde{d}\left(\widetilde{y_\varrho},(T,\alpha)(\widetilde{y_\varrho})\right)^{b}$, where $b\,\tilde{\in}\,[\bar{0},\frac{\bar{1}}{2})$ and $\widetilde{x_\eta},\widetilde{y_\varrho}$ is in $\tilde{X}$

(c) $\tilde{d}\left((T,\alpha)(\widetilde{x_\eta}),(T,\alpha)(\widetilde{y_\varrho})\right)\,\tilde{\leqq}\,\tilde{d}\left(\widetilde{x_\eta},(T,\alpha)(\widetilde{y_\varrho})\right)^{c}.\tilde{d}\left(\widetilde{y_\varrho},(T,\alpha)(\widetilde{x_\eta})\right)^{c}$, where $c\,\tilde{\in}\,[\bar{0},\frac{\bar{1}}{2})$ and $\widetilde{x_\eta},\widetilde{y_\varrho}$ is in $\tilde{X}$

II. Main Results

Lemma 6: Let $\{\widetilde{a_{n,\lambda}}\}$, $\{\widetilde{b_{n,\lambda}}\}$, and $\{\widetilde{t_{n,\lambda}}\}$ be sequences of nonnegative soft numbers satisfying, $\widetilde{a_{n+1,\lambda}}\,\tilde{\leqq}\,\widetilde{a_{n,\lambda}}^{1-\widetilde{\omega_{n,\lambda}}}.\widetilde{b_{n,\lambda}}.\widetilde{t_{n,\lambda}}$, for all $n=0,1,2,\ldots$, where $\{\widetilde{\omega_{n,\lambda}}\}$ is a soft sequence in $[\bar{0},\bar{1}]$.

If $\sum_n\widetilde{\omega_{n,\lambda}}=\bar{\infty}$, $\widetilde{b_{n,\lambda}}=O(\widetilde{\omega_{n,\lambda}})$, and $\sum_n\widetilde{t_{n,\lambda}}\,\tilde{<}\,\bar{\infty}$, then, $\lim\widetilde{a_{n,\lambda}}=\bar{1}$.

Remark 7: The above Lemma is the multiplicative soft analog of a result of Berinde [V. Berinde, Iterative approximation of fixed points, Springer-Verlag Berlin Heidelberg 2007, p.13]

Theorem 8: Let $\tilde{X}$ be a multiplicative soft analog of normed linear space, and let $\tilde{K}$ be a nonempty, closed, multiplicative convex subset of $\tilde{X}$. Let $(T,\alpha):\tilde{K}\to\tilde{K}$ be a multiplicative Z-operator whose set of fixed points is nonempty. Let $(S,\alpha):\tilde{K}\to\tilde{K}$ be continuous and $(S-T,\alpha)$ be bounded. Let $\{\widetilde{x_{n,\lambda}}\}$ satisfy Definition 1 with $\sum_n\widetilde{\delta_{n,\lambda}}=\bar{\infty}$ and $\sum_n\widetilde{c_{n,\lambda}}\,\tilde{<}\,\bar{\infty}$, then $\{\widetilde{x_{n,\lambda}}\}$ converges to a fixed point of $(T,\alpha):\tilde{K}\to\tilde{K}$.

Proof: Since the set of fixed points of $(T,\alpha):\tilde{K}\to\tilde{K}$ is nonempty, then $(T,\alpha):\tilde{K}\to\tilde{K}$ has at least one fixed point in $\tilde{K}$, call it $\widetilde{p_\theta}$. Since $(T,\alpha):\tilde{K}\to\tilde{K}$ is a multiplicative Z-operator, then at least one of (a), (b), (c) of Definition 5 holds. If (b) holds, then from the multiplicative soft norm and multiplicative triangle inequality, we have,

$$\left\|(T,\alpha)(\widetilde{x_\eta})\,\widetilde{-}\,(T,\alpha)(\widetilde{y_\varrho})\right\|\,\tilde{\leqq}\,\left\|\widetilde{x_\eta}-\widetilde{(T,\alpha)}(\widetilde{x_\eta})\right\|^{b}.\left\|\widetilde{y_\varrho}-\widetilde{(T,\alpha)}(\widetilde{y_\varrho})\right\|^{b}\,\tilde{\leqq}$$

$\|\widetilde{x_\eta} - \widetilde{(T,\alpha)}(\widetilde{x_\eta})\|^{b} . \|\widetilde{y_\varrho - x_\eta}\|^{b} . \|\widetilde{x_\eta} - \widetilde{(T,\alpha)}(\widetilde{x_\eta})\|^{b} . \|(T,\alpha)\widetilde{(\widetilde{x_\eta}) - (T,\alpha)}(\widetilde{y_\varrho})\|^{b}$, which implies that,

$\|(T,\alpha)\widetilde{(\widetilde{x_\eta}) - (T,\alpha)}(\widetilde{y_\varrho})\|^{1-b} \precapprox \|\widetilde{y_\varrho - x_\eta}\|^{b} . \|\widetilde{x_\eta} - \widetilde{(T,\alpha)}(\widetilde{x_\eta})\|^{2b}$. Since $b \,\widetilde{\in}\, [\bar{0}, \bar{\frac{1}{2}})$, this further implies that, $\|(T,\alpha)\widetilde{(\widetilde{x_\eta}) - (T,\alpha)}(\widetilde{y_\varrho})\| \precapprox \|\widetilde{y_\varrho - x_\eta}\|^{\frac{b}{1-b}} . \|\widetilde{x_\eta} - \widetilde{(T,\alpha)}(\widetilde{x_\eta})\|^{\frac{2b}{1-b}}$. Similarly, if (c) of Definition 5 holds, then, $\|(T,\alpha)\widetilde{(\widetilde{x_\eta}) - (T,\alpha)}(\widetilde{y_\varrho})\| \precapprox \|\widetilde{y_\varrho - x_\eta}\|^{\frac{c}{1-c}} . \|\widetilde{x_\eta} - \widetilde{(T,\alpha)}(\widetilde{x_\eta})\|^{\frac{2c}{1-c}}$. Put $\xi = max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$, then it follows that,
$\|(T,\alpha)\widetilde{(\widetilde{x_\eta}) - (T,\alpha)}(\widetilde{y_\varrho})\| \precapprox \|\widetilde{y_\varrho - x_\eta}\|^{\xi} . \|\widetilde{x_\eta} - \widetilde{(T,\alpha)}(\widetilde{x_\eta})\|^{2\xi}$. If $\widetilde{x_\eta} = \widetilde{p_\theta}$ and $\widetilde{y_\varrho} = \widetilde{x_{n,\lambda}}$ in
$\|(T,\alpha)\widetilde{(\widetilde{x_\eta}) - (T,\alpha)}(\widetilde{y_\varrho})\| \precapprox \|\widetilde{y_\varrho - x_\eta}\|^{\xi} . \|\widetilde{x_\eta} - \widetilde{(T,\alpha)}(\widetilde{x_\eta})\|^{2\xi}$, then we get,

$\|(T,\alpha)\widetilde{(\widetilde{p_\theta}) - (T,\alpha)}(\widetilde{x_{n,\lambda}})\| \precapprox \|\widetilde{p_\theta - x_{n,\lambda}}\|^{\xi} . \|\widetilde{p_\theta} - \widetilde{(T,\alpha)}(\widetilde{p_\theta})\|^{2\xi}$, but then, $\widetilde{p_\theta}$ is a fixed point of (T,α), thus it follows that , $\|\widetilde{p_\theta} - \widetilde{(T,\alpha)}(\widetilde{x_{n,\lambda}})\| \precapprox \|\widetilde{p_\theta - x_{n,\lambda}}\|^{\xi}$. Since $\{\widetilde{x_{n,\lambda}}\}$ satisfy Definition 1, we notice that,
$\widetilde{x_{n+1,\lambda}} - \widetilde{p_\theta} = \widetilde{a_{n,\gamma}}\widetilde{x_{n,\gamma}} + \widetilde{b_{n,\gamma}}(T,\alpha)(\widetilde{x_{n,\gamma}}) + \widetilde{c_{n,\gamma}}(S,\alpha)(\widetilde{x_{n,\gamma}}) - \widetilde{p_\theta} = (1 - \widetilde{\delta_{n,\lambda}})(\widetilde{x_{n,\lambda}} - \widetilde{p_\theta}) + (\widetilde{\delta_{n,\lambda}} - \widetilde{c_{n,\gamma}})(T,\alpha)(\widetilde{x_{n,\lambda}}) + \widetilde{c_{n,\gamma}}(S,\alpha)(\widetilde{x_{n,\gamma}}) - \widetilde{\delta_{n,\lambda}}\widetilde{p_\theta} = (1 - \widetilde{\delta_{n,\lambda}})(\widetilde{x_{n,\lambda}} - \widetilde{p_\theta}) + \widetilde{\delta_{n,\lambda}}\left((T,\alpha)(\widetilde{x_{n,\gamma}}) - \widetilde{p_\theta}\right) + \widetilde{c_{n,\gamma}}(S - T,\alpha)(\widetilde{x_{n,\lambda}})$

So by multiplicative soft norm, multiplicative triangle inequality, and the fact that
$\|\widetilde{p_\theta} - \widetilde{(T,\alpha)}(\widetilde{x_{n,\lambda}})\| \precapprox \|\widetilde{p_\theta - x_{n,\lambda}}\|^{\xi}$, we deduce,

$$\begin{aligned}\|\widetilde{x_{n+1,\lambda} - p_\theta}\| \,\widetilde{\leq}\, \|\widetilde{x_{n,\lambda} - p_\theta}\|^{1-\widetilde{\delta_{n,\lambda}}} . \|(T,\alpha)\widetilde{(\widetilde{x_{n,\gamma}})} - \widetilde{p_\theta}\|^{\widetilde{\delta_{n,\lambda}}} . \|(S - \widetilde{T,\alpha)}(\widetilde{x_{n,\lambda}})\|^{\widetilde{c_{n,\gamma}}} \precapprox \\ \|\widetilde{x_{n,\lambda} - p_\theta}\|^{1-\widetilde{\delta_{n,\lambda}}} . \|\widetilde{p_\theta - x_{n,\lambda}}\|^{\widetilde{\delta_{n,\lambda}}\xi} . \|(S - \widetilde{T,\alpha)}(\widetilde{x_{n,\lambda}})\|^{\widetilde{c_{n,\gamma}}} \\ = \|\widetilde{x_{n,\lambda} - p_\theta}\|^{1-\widetilde{\delta_{n,\lambda}}+\widetilde{\delta_{n,\lambda}}\xi} . \|(S - \widetilde{T,\alpha)}(\widetilde{x_{n,\lambda}})\|^{\widetilde{c_{n,\gamma}}}\end{aligned}$$

Since $\bar{0} \precapprox \xi \,\widetilde{<}\, \bar{1}$, $\bar{0} \precapprox \widetilde{\delta_{n,\lambda}} \precapprox \bar{1}$, and $\sum_n \widetilde{\delta_{n,\lambda}} = \bar{\infty}$, then with $\widetilde{a_{n,\lambda}} = \|\widetilde{x_{n,\lambda} - p_\theta}\|$ and $\widetilde{\omega_{n,\lambda}} = (1 - \xi)\widetilde{\delta_{n,\lambda}}$, we get the form of Lemma 6, which implies that $\lim\|\widetilde{x_{n,\lambda} - p_\theta}\| = \bar{1}$. So $\{\widetilde{x_{n,\lambda}}\}$ converges to $\widetilde{p_\theta}$, the fixed point of $(T,\alpha): \widetilde{K} \to \widetilde{K}$.

Theorem 9: Let $\widetilde{X}$ be a multiplicative soft analog of normed linear space, and let $\widetilde{K}$ be a nonempty, closed, multiplicative convex subset of $\widetilde{X}$. Let $(T,\alpha), (S,\beta): \widetilde{K} \to \widetilde{K}$ be two multiplicative Z-operators with a common fixed point in $\widetilde{K}$. Let $\{\widetilde{x_{n,\gamma}}\}$ satisfy Definition 3 with $\sum_n(\widetilde{\delta_{n,\lambda}} + \widetilde{\zeta_{n,\lambda}} - \widetilde{\delta_{n,\lambda}}\,\widetilde{\zeta_{n,\lambda}}) = \bar{\infty}$, then $\{\widetilde{x_{n,\gamma}}\}$ converges to the common fixed point of $(T,\alpha), (S,\beta): \widetilde{K} \to \widetilde{K}$.

Proof: Let $\widetilde{p_\theta}$ be the common fixed point of $(T,\alpha), (S,\beta): \widetilde{K} \to \widetilde{K}$. Since both $(T,\alpha), (S,\beta): \widetilde{K} \to \widetilde{K}$ are multiplicative Z-operators, proceeding with arguments similar to those in the proof of previous theorem, we get that $\|(S,\beta)\widetilde{(\widetilde{x_\eta}) - (S,\beta)}(\widetilde{y_\varrho})\| \precapprox \|\widetilde{y_\varrho - x_\eta}\|^{\xi} . \|\widetilde{x_\eta} - \widetilde{(S,\beta)}(\widetilde{x_\eta})\|^{2\xi}$ and

$\left\|(T,\alpha)(\widetilde{x_\eta})\widetilde{-}(T,\alpha)(\widetilde{y_\varrho})\right\| \ \widetilde{\leqq} \ \left\|\widetilde{y_\varrho}\widetilde{-}\widetilde{x_\eta}\right\|^{\xi}.\left\|\widetilde{x_\eta}-(\widetilde{T,\alpha})(\widetilde{x_\eta})\right\|^{2\xi}$. If $\widetilde{x_\eta}=\widetilde{p_\theta}$ and $\widetilde{y_\varrho}=\widetilde{y_{n,\lambda}}$ in $\left\|(S,\beta)(\widetilde{x_\eta})\widetilde{-}(S,\beta)(\widetilde{y_\varrho})\right\| \ \widetilde{\leqq} \ \left\|\widetilde{y_\varrho}\widetilde{-}\widetilde{x_\eta}\right\|^{\xi}.\left\|\widetilde{x_\eta}-(\widetilde{S,\beta})(\widetilde{x_\eta})\right\|^{2\xi}$ and $\left\|(T,\alpha)(\widetilde{x_\eta})\widetilde{-}(T,\alpha)(\widetilde{y_\varrho})\right\| \ \widetilde{\leqq} \ \left\|\widetilde{y_\varrho}\widetilde{-}\widetilde{x_\eta}\right\|^{\xi}.\left\|\widetilde{x_\eta}-(\widetilde{T,\alpha})(\widetilde{x_\eta})\right\|^{2\xi}$, then one concludes that, $\left\|\widetilde{p_\theta}-(\widetilde{T,\alpha})(\widetilde{y_{n,\lambda}})\right\| \ \widetilde{\leqq} \ \left\|\widetilde{p_\theta}\widetilde{-}\widetilde{y_{n,\lambda}}\right\|^{\xi}$ and $\left\|\widetilde{p_\theta}-(\widetilde{S,\beta})(\widetilde{y_{n,\lambda}})\right\| \ \widetilde{\leqq} \ \left\|\widetilde{p_\theta}\widetilde{-}\widetilde{y_{n,\lambda}}\right\|^{\xi}$, since $\widetilde{p_\theta}$ is the common fixed point of $(T,\alpha),(S,\beta):\widetilde{K}\to\widetilde{K}$. Since $\{\widetilde{x_{n,\lambda}}\}$ satisfy Definition 3, we notice that, $\widetilde{x_{n+1,\lambda}}-\widetilde{p_\theta}=\widetilde{a_{n,\gamma}}\widetilde{y_{n,\gamma}}+\widetilde{b_{n,\gamma}}(T,\alpha)(\widetilde{y_{n,\gamma}})+\widetilde{c_{n,\gamma}}(S,\alpha)(\widetilde{y_{n,\gamma}})-\widetilde{p_\theta}=(1-\widetilde{\delta_{n,\lambda}})\widetilde{y_{n,\lambda}}+\widetilde{b_{n,\gamma}}(T,\alpha)(\widetilde{y_{n,\gamma}})+\widetilde{c_{n,\gamma}}(S,\alpha)(\widetilde{y_{n,\gamma}})-(\widetilde{a_{n,\gamma}}+\widetilde{b_{n,\gamma}}+\widetilde{c_{n,\gamma}})\widetilde{p_\theta}=(1-\widetilde{\delta_{n,\lambda}})\widetilde{y_{n,\lambda}}-(1-\widetilde{\delta_{n,\lambda}})\widetilde{p_\theta}+\widetilde{b_{n,\gamma}}\left((T,\alpha)(\widetilde{y_{n,\gamma}})-\widetilde{p_\theta}\right)+\widetilde{c_{n,\gamma}}\left((S,\alpha)(\widetilde{y_{n,\gamma}})-\widetilde{p_\theta}\right)=(1-\widetilde{\delta_{n,\lambda}})(\widetilde{y_{n,\gamma}}-\widetilde{p_\theta})+\widetilde{b_{n,\gamma}}\left((T,\alpha)(\widetilde{y_{n,\gamma}})-\widetilde{p_\theta}\right)+\widetilde{c_{n,\gamma}}\left((S,\alpha)(\widetilde{y_{n,\gamma}})-\widetilde{p_\theta}\right)$. So by the multiplicative soft norm, multiplicative triangle inequality, and the fact that, $\left\|\widetilde{p_\theta}-(\widetilde{T,\alpha})(\widetilde{y_{n,\lambda}})\right\| \ \widetilde{\leqq} \ \left\|\widetilde{p_\theta}\widetilde{-}\widetilde{y_{n,\lambda}}\right\|^{\xi}$ and $\left\|\widetilde{p_\theta}-(\widetilde{S,\beta})(\widetilde{y_{n,\lambda}})\right\| \ \widetilde{\leqq} \ \left\|\widetilde{p_\theta}\widetilde{-}\widetilde{y_{n,\lambda}}\right\|^{\xi}$, we deduce that,

$$\left\|\widetilde{x_{n+1,\lambda}-\widetilde{p_\theta}}\right\| \widetilde{\leqq} \left\|\widetilde{y_{n,\gamma}-\widetilde{p_\theta}}\right\|^{1-\widetilde{\delta_{n,\lambda}}}.\left\|(T,\alpha)\widetilde{(\widetilde{y_{n,\gamma}})-\widetilde{p_\theta}}\right\|^{\widetilde{b_{n,\gamma}}}.\left\|(S,\alpha)\widetilde{(\widetilde{y_{n,\gamma}})-\widetilde{p_\theta}}\right\|^{\widetilde{c_{n,\gamma}}} \widetilde{\leqq} \left\|\widetilde{y_{n,\gamma}-\widetilde{p_\theta}}\right\|^{1-\widetilde{\delta_{n,\lambda}}}.\left\|\widetilde{y_{n,\gamma}-\widetilde{p_\theta}}\right\|^{\widetilde{b_{n,\gamma}}\,\xi}.\left\|\widetilde{y_{n,\gamma}-\widetilde{p_\theta}}\right\|^{\widetilde{c_{n,\gamma}}\,\xi}=\left\|\widetilde{y_{n,\gamma}-\widetilde{p_\theta}}\right\|^{1-\widetilde{\delta_{n,\lambda}}\,(1-\xi)}$$

On the other hand, from Definition 3, we have

$$\widetilde{y_{n,\gamma}}-\widetilde{p_\theta}=(1-\widetilde{\zeta_{n,\gamma}})\,\widetilde{x_{n,\gamma}}+\widetilde{\zeta_{n,\gamma}}(T,\alpha)(\widetilde{x_{n,\gamma}})-\widetilde{p_\theta}=(1-\widetilde{\zeta_{n,\gamma}})(\widetilde{x_{n,\gamma}}-\widetilde{p_\theta})+\widetilde{\zeta_{n,\gamma}}\left((T,\alpha)(\widetilde{x_{n,\gamma}})-\widetilde{p_\theta}\right)$$

Also, since $\left\|\widetilde{p_\theta}-(\widetilde{T,\alpha})(\widetilde{y_{n,\lambda}})\right\| \ \widetilde{\leqq} \ \left\|\widetilde{p_\theta}\widetilde{-}\widetilde{y_{n,\lambda}}\right\|^{\xi}$, we get, $\left\|\widetilde{p_\theta}-(\widetilde{T,\alpha})(\widetilde{x_{n,\lambda}})\right\| \ \widetilde{\leqq} \ \left\|\widetilde{p_\theta}\widetilde{-}\widetilde{x_{n,\lambda}}\right\|^{\xi}$

So by multiplicative soft norm and multiplicative triangle inequality, we have,

$$\left\|\widetilde{y_{n,\gamma}-\widetilde{p_\theta}}\right\| \widetilde{\leqq} \left\|\widetilde{x_{n,\gamma}-\widetilde{p_\theta}}\right\|^{1-\widetilde{\zeta_{n,\gamma}}}.\left\|(T,\alpha)\widetilde{(\widetilde{x_{n,\gamma}})-\widetilde{p_\theta}}\right\|^{\widetilde{\zeta_{n,\gamma}}} \widetilde{\leqq} \left\|\widetilde{x_{n,\gamma}-\widetilde{p_\theta}}\right\|^{1-\widetilde{\zeta_{n,\gamma}}}.\left\|\widetilde{x_{n,\gamma}-\widetilde{p_\theta}}\right\|^{\widetilde{\zeta_{n,\gamma}}\xi}=\left\|\widetilde{x_{n,\gamma}-\widetilde{p_\theta}}\right\|^{1-\widetilde{\zeta_{n,\lambda}}\,(1-\xi)}$$

So, $\left\|\widetilde{x_{n+1,\lambda}-\widetilde{p_\theta}}\right\| \widetilde{\leqq} \left\|\widetilde{y_{n,\gamma}-\widetilde{p_\theta}}\right\|^{1-\widetilde{\delta_{n,\lambda}}\,(1-\xi)} \widetilde{\leqq} \left\|\widetilde{x_{n,\gamma}-\widetilde{p_\theta}}\right\|^{\left(1-\widetilde{\delta_{n,\lambda}}\,(1-\xi)\right)\left(1-\widetilde{\zeta_{n,\lambda}}\,(1-\xi)\right)}=\left\|\widetilde{x_{n,\gamma}-\widetilde{p_\theta}}\right\|^{1-\left(\widetilde{\delta_{n,\lambda}}+\widetilde{\zeta_{n,\lambda}}-\widetilde{\delta_{n,\lambda}}\,\widetilde{\zeta_{n,\lambda}}\right)(1-\xi)}$. Since $\bar{0}\,\widetilde{\leqq}\,\xi\,\widetilde{<}\,\bar{1}$, $\bar{0}\,\widetilde{\leqq}\,\widetilde{\delta_{n,\lambda}}+\widetilde{\zeta_{n,\lambda}}-\widetilde{\delta_{n,\lambda}}\,\widetilde{\zeta_{n,\lambda}}\,\widetilde{\leqq}\,\bar{1}$, and $\sum_n\left(\widetilde{\delta_{n,\lambda}}+\widetilde{\zeta_{n,\lambda}}-\widetilde{\delta_{n,\lambda}}\,\widetilde{\zeta_{n,\lambda}}\right)=\bar{\infty}$, then setting $\widetilde{a_{n,\lambda}}=\left\|\widetilde{x_{n,\gamma}-\widetilde{p_\theta}}\right\|$ and

$\widetilde{\omega_{n,\lambda}}=\left(\widetilde{\delta_{n,\lambda}}+\widetilde{\zeta_{n,\lambda}}-\widetilde{\delta_{n,\lambda}}\,\widetilde{\zeta_{n,\lambda}}\right)(1-\xi)$, we get the form of Lemma 6, which implies that $\lim\left\|\widetilde{x_{n,\lambda}-\widetilde{p_\theta}}\right\|=\bar{1}$. So $\{\widetilde{x_{n,\lambda}}\}$ converges to $\widetilde{p_\theta}$, the common fixed point of $(T,\alpha),(S,\beta):\widetilde{K}\to\widetilde{K}$, and the proof is finished.

Remark 10: When $(T,\alpha)=(S,\beta)$ in Definition 1, then we get a Corollary to Theorem 8 and Theorem 9, respectively.

Question 11: Consider Theorem 2.1 of Berinde [V. Berinde, On the convergence theorem for Mann iteration in the class of Zamfirescu operators, Analele Universitatii de Vest. Timisoara Seria Matematica-infomatic XLV.1(2007), 33-41] to the setting of this paper. In light of Remark 10, in what way is it a corollary to the results obtained in this paper? Similarly, consider Theorem 2.1 of Yildirim [I. Yildirim, M.Ozdemir, H.Kiiziltunc, On convergence of a new step iteration in the class of Quasi contractive operators. Int. Journal of Math.Analysis. 3(2009), 1881-1892] to the setting of this paper. In light of Remark 10, in what way is it a corollary to the results obtained in this paper?

References

[1] S. Thianwan, Common fixed points of the new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space, J.Comput Appl. Math. 224(2009), 688-695

[2] V. Berinde, Iterative approximation of fixed points, Springer-Verlag Berlin Heidelberg 2007, p.13

[3] V. Berinde, On the convergence theorem for Mann iteration in the class of Zamfirescu operators, Analele Universitatii de Vest. Timisoara Seria Matematica-infomatic XLV.1(2007), 33-41

[4] I. Yildirim, M.Ozdemir, H.Kiiziltunc, On convergence of a new step iteration in the class of Quasi contractive operators. Int. Journal of Math.Analysis. 3(2009), 1881-1892