FDT FOR Ω-MONOIDS

Jollanda Shara

Department of Mathematics&Computer Science, University "Eqrem Cabej", 6001, Gjirokaster, Albania.

Email: jokrisha@yahoo.com

Abstract

In this paper we generalize the results of C.Squier ([1]) in the case of Ω-monoids. We give, first, the definition of Ω-semigroups and some general results related to the Ω-string rewriting systems, the properties of confluence, termination, Church-Rosser, and so on. Finally, we prove our main theorem which states that if M is a finitely presented Ω-monoid which has a presentation $(X; R)$ involving a finite convergent Ω-string rewriting system R, then M has finite derivation type.

Keywords: **Ω-semigroup, FDT, string-rewriting systems, derivation graph, homotopy.**

1. Introduction.

In the recent years string-rewriting systems have been the central theme of numerous important papers in theoretical computer science and mathematics in general. Now, it is a well known the fact that if a monoid can be presented by a finite and complete (that is, noetherian and confluent) string-rewriting system, then the word problem for this monoid is solvable. Unfortunately, the property of having a finite and complete string-rewriting system is not invariant from the given presentation. But, for finitely presented monoids, there exists another finiteness condition which is introduced by Squier (see [1]) and is called finite derivation type (or FDT, for short). It is obtained by considering certain binary relations, called homotopy relations, on the set of paths of the derivation graph (the so called Squier's complex) that is associated with a finite presentation $(X; R)$ of the monoid M considered. A monoid has FDT if the full homotopy relation is generated by a finite set called a homotopy base. Squier proved that this property is indeed a property of finitely presented monoids, that is, it is an intrinsic property of a monoid independent of its presentation. He established the fact that every monoid that can be presented through a finite convergent presentation does have FDT. Thus, FDT is one of the necessary conditions that a finitely presented monoid must satisfy in order that it can be presented by some finite convergent string-rewriting system.

In this paper we generalize these results in the case of Ω-monoids. We define, first, the Ω-semigroups as a universal algebra which is a semigroup and in which there is given a system of binary operations Ω satisfying the associative condition:

$$((x,y)\alpha,z)\beta = (x,(y,z)\beta)\alpha$$

for all $x,y,z \in S$ and for each pair of binary operations α,β.

In the first sections of the paper we define and give some general results related to the Ω-string rewriting systems, the properties of confluence, Noetherian, Church-Rosser, critical peaks, the word problem for the Ω-monoids and so on. The last two sections are dedicated to the property of finite derivation type (FDT) and the related results of Squier ([1]) generalized in the case of Ω-monoids. We prove here our main theorem which states that if M is a finitely presented Ω-monoid which has a presentation $(X;R)$ involving a finite convergent Ω-string rewriting system R, then M has finite derivation type.

2. Preliminaries.

A binary relation on X is a subset $R \subseteq X \times X$. If $(x,y) \in R$, then we denote this by xRy and we say that x is related to y by R. The inverse relation of R is the binary relation $R^{-1} \subseteq X \times X$ defined by $yR^{-1}x \Leftrightarrow (x,y) \in R$. The relation $I_X = \{(x,x), x \in X\}$ is called the identity relation. The relation $(X)^2$ is called the complete relation.

Let $R \subseteq X \times X$ and $S \subseteq X \times X$ two binary relations. The composition of R and S is a binary relation $S \circ R \subseteq X \times X$ defined by $xS \circ Rz \Leftrightarrow \exists y \in X$ such that xRy and ySz.

A binary relation R on a set X is said to be

1. Reflexive if xRx for all $x \in X$;

2. Symmetric if xRy implies yRx;
3. Transitive if xRy and yRz imply xRz;
4. Antisymmetric if xRy and yRx imply $x = y$.

Let R be a relation on a set X. The reflexive closure of R is the smallest reflexive relation R^0 on X that contains R; that is,

1. $R \subseteq R^0$
2. If R' is a reflexive relation on X and $R \subseteq R'$, then $R^0 \subseteq R'$.

The symmetric closure of R is the smallest symmetric relation R^+ on X that contains R; that is

1. $R \subseteq R^+$
2. If R' is a symmetric relation on X and $R \subseteq R'$ then $R^+ \subseteq R'$.

The transitive closure of R is the smallest transitive relation R^* on X that contains R; that is

1. $R \subseteq R^*$
2. If R' is a transitive relation on X and $R \subseteq R'$ then $R^* \subseteq R'$.

Let R be a relation on a set X. Then

1. $R^0 = R \cup I_X$
2. $R^+ = R \cup R^{-1}$
3. $R^* = \bigcup_{k=1}^{k=+\infty} R^k$.

Let X be an alphabet. A semi-Thue system R over X, for briefly STS, is a finite set $R \subseteq X^* \times X^*$, whose elements are called rules. A rule (s, t) will also be written as $s \rightarrow t$. The set $dom(R)$ of all left-hand sides and $ran(R)$ of all right-hand sides are defined as follows:

$dom(R) = \{s \in X^*, \exists t \in X^*: (s, t) \in R\}$ and

$ran(R) = \{t \in X^*, \exists s \in X^*: (s, t) \in R\}$.

If R is finite, then the size of R is denoted by $||R||$ and is defined as

$$\left||R|\right| = \sum_{(s,t)\in R} (|s| + |t|).$$

We define the binary relation $\rightarrow_R$ as follows, where $u, v \in X^*$:

$u \rightarrow_R v$ if there exist $x, y \in X^*$ and $(r, s) \in R$ with $u = xry$ and $v = xsy$. We write $u \rightarrow_R^* v$ if there are words $u_0, u_1, \dots, u_n \in X^*$ such that $u_0 = u, u_i \rightarrow_R u_{i+1}, \forall 0 \leq i \leq n-1, u_n = v$.

If $n = 0$, we have $u = v$, and if $n = 1$, then we have $u \rightarrow_R v$.

Note that $\rightarrow_R^*$ is the reflexive transitive closure of $\rightarrow_R$. The Thue congruence $\leftrightarrow_R^*$ is the equivalence relation generated by $\rightarrow_R$. If R is a relation on X^* and $R^\#$ denotes the congruence generated by R then the relations $\leftrightarrow_R^*$ and $R^\#$ coincide.

A decision problem is a restricted type of an algorithmic problem where for each input there are only two possible outputs. In other words, a decision problem is a function that associates with each input instance of the problem a truth value true or false.

Definition 2.2. A graph G is a 5-tuple $G = (V, E, \sigma, \tau, ^{-1})$, where V is the set of vertices and E is the set of edges of G; $\sigma, \tau: E \rightarrow V$ are mappings, which associate with each edge $e \in E$ its initial vertex $\sigma(e)$ and its terminal vertex $\tau(e)$, respectively.;and $^{-1}: E \rightarrow E$ is a mapping satisfying the following conditions: $e^{-1} \neq e, (e^{-1})^{-1} = e, \sigma(e^{-1}) = \tau(e)$ and $\tau(e^{-1}) = \sigma(e)$ for all $e \in E$.

Definition 2.3. Let $G = (V, E, \sigma, \tau, ^{-1})$ be a graph, and let $n \in \mathbb{N}$. A path in G (of length n) is a $(2n+1)$-tuple $p = (v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n)$ with $v_0, v_1, \dots, v_n \in V$ and $e_1, e_2, \dots, e_n \in E$ such that $\sigma(e_i) = v_{i-1}$ and $\tau(e_i) = v_i$ hold for all $i = 1,2, \dots, n$. In this situation p is a path from v_0 to v_n, and the mappings σ, τ can be extended to paths by setting $\sigma(p) = v_0$ and $\tau(p) = v_n$. For $u, v \in V$, $P(u, v)$ denotes the set of paths in G from u to v. In particular, for each $v \in V$, $P(v, v)$ contains the empty path (v).

Also the mapping $^{-1}$ can be extended to paths. The inverse path $p^{-1} \in P(v_n, v_0)$ of p is the following path $p^{-1} = (v_n, e_n^{-1}, v_{n-1}, \dots, v_1, e_1^{-1}, v_0)$. Finally, if $p \in P(u, v)$ and $q \in P(v, w)$, then the composite path $p \circ q \in P(u, w)$ is defined in the obvious way.

It is clear that, the composition of paths is an associative operation, and the empty paths act as identities for composition. Next, if $p \in P(u,v)$, then $(p^{-1})^{-1} = p$, and if $q \in P(v,w)$ then $(p \circ q)^{-1} = q^{-1} \circ p^{-1}$. Finally, if p is an empty path, then $p^{-1} = p$.

If G is a graph, then $P(G)$ will denote the set of all paths in G, and $P^{(2)}(G) = \{(p,q)|p,q \in P(G) such\ that\ \sigma(p) = \sigma(q) and\ \tau(p) = \tau(q)\}$ is the set of all pairs of paths that have a common initial vertex and a common terminal vertex.

Definition 2.4. Let $G_1 = (V_1, E_1, \sigma_1, \tau_1, ^{-1})$ and $G_2 = (V_2, E_2, \sigma_2, \tau_2, ^{-1})$ be graphs. A mapping from G_1 to G_2 is an ordered pair $f = (f_V, f_E)$ of functions, where $f_V: V_1 \to V_2$ and for each $e \in E_1$, $f_E(e)$ is a path in G_2 from $f_V(\sigma_1(e))$ to $f_V(\tau_1(e))$. Further, for each $e \in E_1$, $f_E(e^{-1}) = (f_E(e))^{-1}$. The mapping f is called a morphism if f_E carries edges to edges.

It is clear that a mapping $f: G_1 \to G_2$ induces a mapping $f: P(G_1) \to P(G_2)$.

Definition 2.5. Let $G = (V, E, \sigma, \tau, ^{-1})$ be a graph. A subgraph $G_1 = (V_1, E_1, \sigma_1, \tau_1, ^{-1})$ of G consists of a subset V_1 of V and a subset E_1 of E such that, for all $e \in E_1$, $\sigma_1(e) = \sigma(e) \in V_1$ and $\tau_1(e) = \tau(e) \in V_1$. Next, $e^{-1} \in E_1$ for all $e \in E_1$.

Definition 2.6.([6]) A type of universal algebras is an ordered pair of a set T and a mapping $\omega \longmapsto n_\omega$ that assigns to each $\omega \in T$ a nonnegative integer n_ω, the formal arity of ω. A universal algebra, or just algebra of type T is an ordered pair of a set A and a mapping, the type – T algebra structure on A, that assigns to each $\omega \in T$ an operation ω_A on A of arity n_ω.

3. Presentations of Ω-monoids.

A semigroup with multiple operators or a Ω-semigroup is a universal algebra which is a semigroup and in which there is given a system of binary operations Ω satisfying the associative condition:

$$((x,y)\alpha, z)\beta = (x, (y,z)\beta)\alpha$$

for all $x, y, z \in S$ and for each pair of binary operations α, β.

Let (S, Ω), (T, Ω) be two Ω-semigroups. Then, $f: S \to T$ is a homomorphism if

$$f\big((x,y)\omega\big) = \big(f(x), f(y)\big)\omega, x, y \in S, \forall \omega \in \Omega$$

Next, we define the free Ω-semigroup using the concept of the free word algebra of a type T with the set X as basis, as it is described in [6]. For the case of Ω-semigroups, we agree, first, that their type is simply a set of binary relations which we denote by Ω. So, we construct, inductively, the free Ω-word algebras as follows: denote $W_0 = X$, then for $k > 0$ denote W_k the set of all sequences (γ, w_1, w_2) where $w_1, w_2 \in W_{k-1}$ and $\gamma \in \Omega$. For each $\alpha \in \Omega$, we denote by λ_α the empty word related to α. Now, we take $W_X = \bigcup_{k\geq 0} W_k$. Writing this in letters, we will have that W_1 is the set of all sequences (γ, x, y) where $\gamma \in \Omega$ and $x, y \in X$. It is more convenient to denote these sequences in the form $x\gamma y$. The product $x\beta\lambda_\beta$ is defined to be x, and similarly the product of the form $\lambda_\alpha \alpha y$ is defined to be y, where $\lambda_\alpha, \lambda_\beta$ are the empty words related to the operators α, β, respectively. In the next step, W_2 would have as elements the sequences (γ, w_1, w_2) where $w_1, w_2 \in W_1$ and $\gamma \in \Omega$. If $w_1 = x_1\gamma_1 y_1$ and $w_2 = x_2\gamma_2 y_2$, then (γ, w_1, w_2) would be just the sequence $x_1\gamma_1 y_1 \gamma x_2 \gamma_2 y_2$, with our new notations. And this procedure continues …

Examples:

1. A semigroup is a set with a single binary operation . Here Ω consists of a single element μ of arity two such that the following associative law is satisfied $xy\mu z\mu = xyz\mu\mu$ for all $x, y, z \in S$.
2. A Γ-semigroup is a special case of an Ω-semigroup. Indeed, we define in S binary operators $\bar{\alpha}: S \times S \to S$ such that $\bar{\alpha}(x, y) = x\alpha y, \forall \alpha \in \Gamma$. Then, $(S, \bar{\Gamma})$ is a Ω-algebra where $\bar{\Gamma} = \{\bar{\gamma}: \gamma \in \Gamma\}$ satisfying the conditions $\bar{\beta}(\bar{\alpha}(x, y), z) = \bar{\alpha}\left(x, \bar{\beta}(y, z)\right), \forall x, y, z \in S, \bar{\alpha}, \bar{\beta} \in \bar{\Gamma}$.
3. It is clear that the free Ω-semigroup defined as above is a Ω-semigroup.

We will denote with $MX^*\Omega$ the free Ω-monoid on X, that is the set of finite products $x_1\gamma_1 \dots x_{n-1}\gamma_{n-1}x_n$ with $x_1, \dots, x_n \in X, \gamma_i \in \Omega, \mathrm{i} = 1,2, \dots, \mathrm{n} - 1$, including the empty product 1.

It is the smallest Ω-submonoid of M containing X.

If $MX^*\Omega = M$, we say that X generates M, or that X is a set of generators for M. If X is finite and generates M, we say that M is a finitely generated Ω-monoid. If X generates M and no strict subset of X does, we say that X is a minimal set of generators for M.

Proposition 3.1. If M is a finitely generated Ω-monoid and X is a set of generators for M, then there is a finite subset of X which generates M. In particular, any minimal set of generators for M is finite.

Proof:Indeed, for any $y = x_1\gamma_1 \dots x_{n-1}\gamma_{n-1}x_n \in M$ with $x_1, \dots, x_n \in X, \gamma \in \Omega$, we get a finite set $X(y) = \{x_1, \dots, x_n\} \subset X$. If $Y = \{y_1, \dots, y_m\}$ generates M, so does the finite set $X(Y) = X(y_1) \cup \dots \cup X(y_m) \subset X$.

Now, if M is a Ω-monoid, then any map $f: X \to M$ extends to a unique morphism $\bar{f} : MX^*\Omega \to M$.

A presentation (by generators and relations) is a pair $(X; R)$ where X is an alphabet and R is the following set $R = \{(u, v)|\ u, v \in W_X\}$. The congruence generated by R is defined as follows:

i. $u\alpha u'\beta v \leftrightarrow_R u\alpha v'\beta v$ whenever $u, v \in MX^*\Omega, \alpha, \beta \in \Omega$, and $u'Rv'$
ii. $x \leftrightarrow_R^* y$ whenever $x = x_0 \leftrightarrow_R x_1 \leftrightarrow_R \dots \leftrightarrow_R x_n = y$.

We denote by M_R the quotient $M_R = MX^*\Omega/\leftrightarrow_R^*$ which is a Ω-semigroup.

Indeed, it easily verified that the congruence generated by R, as we defined it, is a Ω-congruence. For this, it's enough to see that $u\alpha u'\beta v \leftrightarrow_R u\alpha v'\beta v \implies u\alpha u'\beta v\gamma w \leftrightarrow_R u\alpha v'\beta v\gamma w$ and $u\alpha u'\beta v \leftrightarrow_R u\alpha v'\beta v \implies w\gamma u\alpha u'\beta v \leftrightarrow_R w\gamma u\alpha v'\beta v$. Let us denote shortly by ρ this congruence. Now, for $u\rho, v\rho \in M_R$ and $\gamma \in \Omega$, let $(u\rho)\gamma(v\rho) = (u\gamma v)\rho$. This is well-defined, since for all $u, v \in MX^*\Omega$ and $\gamma \in \Omega$,

$$u\rho = u'\rho \text{ and } v\rho = v'\rho \implies (u, u'), (v, v') \in \rho \implies (u\gamma v, u'\gamma v), (u'\gamma v, u'\gamma v') \in \rho$$

$$\implies (u\gamma v, u'\gamma v') \in \rho \implies (u\gamma v)\rho = (u'\gamma v')\rho$$

Let $u, v, w \in MX^*\Omega$ and $\gamma, \mu \in \Omega$. Then, it follows that

$$(u\rho\gamma v\rho)\mu w\rho = \big((u\gamma v)\rho\big)\mu w\rho = \big((u\gamma v)\mu w\big)\rho = \big(u\gamma(v\mu w)\big)\rho = u\rho\gamma(v\mu w)\rho = u\rho\gamma(v\rho\mu w\rho)$$

and this result completes the proof.

We have a canonical surjection $\pi_R: MX^*\Omega \to MX^*\Omega/\leftrightarrow_R^*$ as well. Moreover, if $f: X \to M$ is a map such that $f(x) = f(y)$ whenever xRy and $\bar{f} : MX^*\Omega \to M$ its extension we obtain a unique morphism $\tilde{f}: MX^*\Omega/\leftrightarrow_R^* \to M$ such that $\tilde{f} \circ \pi_R = \bar{f}$. If the map $\tilde{f}$ is bijective, we write $M \cong MX^*\Omega/\leftrightarrow_R^*$ and we say that $(X; R)$ is a presentation of the Ω-monoid M. This means that the set $f(X)$ generates M, and that $\bar{f}(x) = \bar{f}(y)$ if and only if $x \leftrightarrow_R^* y$. If the map $\tilde{f}$ is bijective and both X and R are finite we say that M is a finitely presented Ω-monoid. And again, if the map $\tilde{f}$ is bijective, $f(X)$ is a minimal set of generators and no strict subset of R generates the congruence $\leftrightarrow_R^*$, then we say that $(X; R)$ is a minimal presentation of M.

Lemma 3.2. For any morphism $: MX^*\Omega/\leftrightarrow_R^* \to MY^*\Omega/\leftrightarrow_S^*$, there is a morphism $\varphi: MX^*\Omega \to MY^*\Omega$ such that $\pi_S \circ \varphi = f \circ \pi_R$.

Proof:

$$\begin{array}{ccc} MX^*\Omega & \overset{\varphi}{\to} & MY^*\Omega \\ \pi_R \downarrow & & \downarrow \pi_S \\ MX^*\Omega/\leftrightarrow_R^* & \underset{f}{\to} & MY^*\Omega/\leftrightarrow_S^* \end{array}$$

It is sufficient to define $\varphi(x)$ for each $x \in X$, and for this we have to use the fact that π_S is surjective.

4. Rewrite rules and reductions.

If $(X; R)$ is a presentation of a Ω-monoid , each $\rho = (x, y) \in R$ can be seen as a rewrite rule $x \overset{\rho}{\to} y$, with source x and target y. An elementary reduction is of the form $u\alpha x\beta v \xrightarrow{u\rho v} u\alpha y\beta v$ where u, v are words and $x \overset{\rho}{\to} y$ is a rule (as we define it) . A reduction is a finite sequence

$$x = x_0 \overset{r_1}{\to} x_1 \overset{r_2}{\to} x_2 \dots x_{n-1} \overset{r_n}{\to} x_n = y$$

of elementary reductions. Each rule is considered as an elementary reduction, and any elementary reduction is considered as a reduction of length 1. If $x \overset{r}{\to} y$ and $y \overset{s}{\to} z$ are reductions, we write $r * s$ for the composed reduction $x \overset{r}{\to} y \overset{s}{\to} z$. Furthermore, there is an empty reduction $x \overset{x}{\to} x$ for any word $x \in MX^*\Omega$. So we obtain a category of reductions $\boldsymbol{C}(X; R)$. We call R a Ω-string rewriting system.

5. Termination and confluence.

The reduction relation generated by R is the smallest order relation containing R which is compatible with the product:

$u\alpha x\beta v \underset{R}{\to} u\alpha y\beta v$ whenever $u, v \in MX^*\Omega$ and xRy in the sense of definition we give for R;

$x \to_R^* y$ whenever $x = x_0 \to_R x_1 \to_R \dots \to_R x_n = y$.

In other words $x \to_R^* y$ whenever there is a reduction $x \overset{r}{\to} y$ and $x \to_R y$ whenever there is an elementary one.

We say that a word x is reducible if there is some word y such that $x \to_R y$. Otherwise we say that x is reduced (irreducible). We denote by $IRR(R)$ the set of irreducible words mod$R$. We say that a property is R-hereditary if, whenever it holds for each y such that $x \to_R y$ then it also holds for x. In particular, such a property holds for all reduced words.

Proposition 5.1. For any presentation $(X; R)$ the following properties are equivalent:

i. There is no infinite reduction $x_0 \to_R x_1 \to_R \dots \to_R x_n \to_R \dots$ (termination);
ii. Any R-hereditary property holds for all words (noetherian induction property).

Proof: If x does not satisfy some R-hereditary property, then we can build an infinite reduction starting from x. Indeed, if $P(x)$ does not hold for some $x \in X$ then by our supposition there will be some $x_1 \in X$ such that $x \to_R x_1$ and $P(x_1)$ is false. Continuing this argument we obtain an infinite sequence $x \to_R x_1 \to_R x_2 \to_R$ .... But this is a contradiction to our assumption that the termination property holds. Conversely, termination can be proved by noetherian induction.

In that case, we say that that the presentation is noetherian. This implies that the source of a rule can never be empty. Moreover, for any word x, there is a reduced word x' such that $x \to_R^* x'$.

In order to prove that a presentation $(X; R)$ is noetherian, it suffices to define a termination ordering for it, that is a strict well-founded ordering $\prec$ which contains R and which is compatible with the product. For instance, $\prec$ may be defined by $x \prec y \Leftrightarrow |x| < |y|$ (length-lexicographical order). We give below the definitions of three other properties, as well:

Church-Rosser property:

If $x \leftrightarrow_R^* y$, there is z such that $x \to_R^* z$ and $y \to_R^* z$.

Confluence:

If $x \to_R^* y$ and $x \to_R^* z$, there is t such that $y \to_R^* t$ and $z \to_R^* t$.

Local confluence:

If $x \to_R y$ and $x \to_R z$, there is t such that $y \to_R^* t$ and $z \to_R^* t$.

A noetherian presentation which satisfies one of the above properties is called convergent (complete).

6. Critical peaks.

As a first step, we define the derivations for the presentation as follows:

1) An atomic derivation $r \xrightarrow{A} s$ is given by a pair $(r, s) \in R$,
2) An elementary derivation $x \xrightarrow{E} y$ is given by two words $u, v \in MX^*\Omega$ and an atomic derivation $r \xrightarrow{A} s$ such that $x = u\alpha r\beta v$ and $y = u\alpha s\beta v$. If $u = v = 1$, we identify E with the atomic derivation A,

3) A derivation $x \xrightarrow{F} y$ is given by a sequence $x = x_0 \xrightarrow{E_1} x_1 \xrightarrow{E_2} \dots \xrightarrow{E_n} x_n = y$ of elementary derivations. If $n = 1$, we identify F with the elementary derivation E_1. If $n = 0$, we get the identity derivation.

Composition of derivations is defined in obvious way. Also, if x, y are words and $z \xrightarrow{F} z'$ is a derivation, the derivation $x\alpha z\beta y \xrightarrow{xFy} x\alpha z'\beta y$ is defined in the obvious way.

Let $(X; R)$ be a Ω-monoid presentation such that the Ω-string-rewriting system R is noetherian. This means that there is no infinite sequence $x_0 \xrightarrow{E_1} x_1 \xrightarrow{E_2} \dots \xrightarrow{E_n} x_n \xrightarrow{E_{n+1}} \dots$ of elementary derivations. Then for any $\in MX^*\Omega$, there is a derivation $x \xrightarrow{F} y$ where y is reduced which means that no elementary derivation starts from y. This y is called a normal form of x.

A peak is an unordered pair of elementary derivations $x \xrightarrow{E} y$ and $x \xrightarrow{E'} y'$ starting from the same word x. Such a peak is called confluent if there is a word z and two derivations $y \xrightarrow{F} z$ and $y' \xrightarrow{F'} z$.

It is called critical if $E \neq E'$ and if it is of the form

$$\begin{array}{ccc} & r\alpha v = u'\alpha' r' & \\ Av \swarrow & & \searrow u'A' \\ s\alpha v & & u'\alpha' s' \end{array}$$

$$\begin{array}{ccc} & u\alpha r\beta v = r' & \\ u\alpha A\beta v \swarrow & & \searrow A' \\ u\alpha s\beta v & & s' \end{array}$$

where, in the first case, u' is a strict prefix of r, or equivalently, v is a strict suffix of r'.

Remarks 6.1.:

- If $p = (r, s)$ is a confluent peak then $q = (s, r)$ is again a confluent peak. Thus we can identify q with p.
- If u is a word and $p = (r, s)$ is a confluent peak then $u\alpha p = (u\alpha r, u\alpha s)$ and $p\alpha u = (r\alpha u, s\alpha u)$, $\alpha \in \Omega$ are also confluent peaks.
- If $x \overset{r}{\rightarrow} y$ and $z \overset{s}{\rightarrow} t$ are elementary reductions then $p = (rz, xs)$ is a confluent peak. In the latter case, we say that the elementary reductions (elementary derivations) rz and xs are disjoint.

So, a peak is critical if it is not of the form $u\alpha p$ or $p\alpha u$ with $u \neq 1$, and if its reductions are neither equal nor disjoint.

A pair of positive edges with the same initial vertex form a critical peak if either:

i. One of the pair is both left- and right-reduced (a critical peak of inclusion type) i.e. it has the form $u_1\alpha' u_2\beta' u_3$ where $u_1, u_2, u_3 \in MX^*\Omega, \alpha', \beta' \in \Omega$.
ii. One of the pair is left-reduced but not right-reduced, the other is right-reduced but not left-reduced, and they are not disjoint (a critical peak of overlapping type), i.e. it has the form $u\beta v\gamma w$ where $u, v, w \in MX^*\Omega, \beta, \gamma \in \Omega$. In this case, $u\beta v$ and $v\gamma w$ are the first coordinates of two pairs from R.

A critical peak of a Ω- string rewriting system R is a pair of overlapping rules if

1) $(u\beta v, s_1), (v\gamma w, s_2) \in R, v \neq \lambda, u, v, w \in MX^*\Omega, \beta, \gamma \in \Omega$; and a pair of inclusion type if
2) $(u\beta v\gamma w, s_1), (v, s_2) \in R, u, v, w \in MX^*\Omega, \beta, \gamma \in \Omega$.

Also a critical peak is resolved in R if there is a word z such that $s_1 w\delta \overset{*}{\rightarrow} z$ and $us_2 \overset{*}{\rightarrow} z$ in the first case or $s_1 \overset{*}{\rightarrow} z$ and $us_2 w \overset{*}{\rightarrow} z$.

The following theorem is a generalization of Theorem 2.1.,[2].

Theorem 6.2. If the presentation $(X; R)$ of a Ω-monoid is noetherian, then the following properties are equivalent:

i. R is Church-Rosser;

ii. If $(u\beta v, s_1), (v\gamma w, s_2) \in R, u, v, w \in MX^*\Omega, \beta, \gamma \in \Gamma$ with $v \neq \lambda$, then there exists $z \in MX^*\Omega$ such that $s_1 w \overset{*}{\rightarrow} z$ and $us_2 \overset{*}{\rightarrow} z$. If $(u\beta v\gamma w, s_1), (v, s_2) \in R, u, v, w \in MX^*\Omega, \beta, \gamma \in \Omega$ then there exists $z \in MX^*\Omega$ such that $s_1 \overset{*}{\rightarrow} z$ and $u\beta s_2 \gamma w \overset{*}{\rightarrow} z$;

iii. For each $x \in MX^*\Omega$ there exists a unique irreducible $z \in MX^*\Omega$ such that $x \overset{*}{\rightarrow} z$.

Proof: From i. follows immediately ii. since $s_1 w \overset{*}{\leftrightarrow} us_2$. ii. implies iii.

If x is reduced, then x itself is the unique reduced form of x. In general, suppose that $x \rightarrow y_i \overset{*}{\rightarrow} z_i$ where z_1, z_2 are irreducible. The relation applications involved in $x \rightarrow y_i, i = 1,2$ are either identical or disjoint or ii. applies. In any case, there exists $y \in MX^*\Omega$ such that $y_1 \overset{*}{\rightarrow} y$ and $y_2 \overset{*}{\rightarrow} y$. Choose an irreducible $z \in MX^*\Omega$ such that $y \overset{*}{\rightarrow} z$. Thus $y_i \overset{*}{\rightarrow} z$ for each $i = 1,2$. Applying the inductive hypothesis twice, each $z_i = z$ which implies $z_1 = z_2$, as required.

It is obvious that iii. implies i. Note first that if iii. holds and $u \rightarrow v$, then u and v have the same irreducible; i. follows immediately if we apply induction on the length of a relation chain which connects x and y in the definition of the Church-Rosser property.

Proposition 6.3. If all the critical peaks of a presentation are confluent then all the peaks are confluent.

Proof: It follows directly from the above remarks.

Corollary 6.3.1. If a presentation is noetherian and all its critical peaks are confluent , then it is convergent.

A noetherian presentation is called complete or canonical if all critical peaks are confluent. The above proposition, implies the confluence of all peaks and the uniqueness of normal forms. Thus, if this presentation is finite, the word problem is decidable.

7. Decision problems.

If $(X; R)$ is a convergent presentation, we denote by $\hat{x}$ the reduced form of x, that is the unique x' such that $x \rightarrow_R^* x'$. By Church-Rosser, we have $x \leftrightarrow_R^* y$ if and only if $\hat{x} = \hat{y}$.

Consider the following decision problem:

INSTANCE: Two words $u, v \in MX^*\Omega$;

QUESTION: Does $u \leftrightarrow_R^* v$ hold?

If this problem is decidable we say that the relation $\leftrightarrow_R^*$ is decidable.

Proposition 7.1. If $(X; R)$ is a finite convergent presentation then $\leftrightarrow_R^*$ is a decidable relation.

Proof: It would be enough to compare the reduced form which, in this case, are obviously computable.

If $\leftrightarrow_R^*$ is a decidable relation then we say that that the Ω-monoid M has a decidable word problem and this property does not depend on the choice of the presentation as long as this presentation is finitely generated, i.e. X is finite. Indeed, assume that $(X; R)$ and $(Y; S)$ are finitely generated presentations of the Ω- monoid M such that $M_R \cong M \cong M_S$. Then for every $a \in X$ there exists a word $w_a \in MY^*\Omega$ such that a and w_a represent the same element of M. If we define the homomorphism $h: MX^*\Omega \rightarrow MY^*\Omega$ by $h(a) = w_a$ then for all $u, v \in MX^*\Omega$ we have $u \leftrightarrow_R^* v$ if and only if $h(u) \leftrightarrow_R^* h(v)$. Thus the word problem for $(X; R)$ can be reduced to the word problem for $(Y; S)$ and vice versa. Thus the decidability and complexity of the word problem does not depend on the chosen presentation. Hence, we may just speak of the word problem for the Ω-monoid M.

Proposition 7.2. Convergence is a decidable property for any finite noetherian presentation.

Proof: It follows from the facts that there are finitely many critical peaks in this case and is easily seen that they are computable.

8. Reduced presentations.

We say that a convergent presentation $(X;R)$ is reduced if each symbol $z \in X$ is reduced, and for each rule $x \xrightarrow{\rho} y$ the source x is only reducible by ρ, whereas the target y is reduced. So we can identify the rule ρ with its source x. Moreover, each critical peak p is an overlap and is determined by its source x. So we can identify again p with x.

Proposition 8.1. For any convergent presentation, there is a reduced one with no more symbols and rules.

Corollary 8.1.1. If M has a finite convergent presentation, then it has a finite reduced convergent presentation.(see [4])

9. Finite derivation type (FDT).

In the following sections we generalize the results of [1].

Let us first give some background material about monoid presentations, associated graphs and the property of finite derivation type. So suppose that $(X;R)$ is a Ω-monoid presentation as we defined it in the previous sections. The Ω-monoid defined by $(X;R)$, as we saw, is the quotient of $MX^*\Omega$ by the smallest congruence generated by R, where R is a Ω-string rewriting system. In fact, we have a graph $G_\Omega = G_\Omega(X;R) = (V, E, \sigma, \tau, {}^{-1})$ associated with $(X;R)$ as follows:

a) $V = MX^*\Omega$ is the set of vertices,
b) $E = \{(u, \alpha, l, r, \beta, v) | u, v \in MX^*\Omega, \alpha, \beta \in \Omega \ and \ (l, r) \in R \cup R^{-1}\}$ is the set of edges,
c) the mappings $\sigma, \tau: E \to V$ are defined through $\sigma(u, \alpha, l, r, \beta, v) = u\alpha l\beta v$ and $\tau(u, \alpha, l, r, \beta, v) = u\alpha r\beta v$, $\alpha, \beta \in \Omega$,
d) the mapping ${}^{-1}: E \to E$ is defined through $(u, \alpha, l, r, \beta, v)^{-1} = (u, \alpha, r, l, \beta, v)$.

If $e = (u, \alpha, l, r, \beta, v)$ is an edge of G_Ω and $x, y \in MX^*\Omega$, then $x\gamma e\delta y = (x\gamma u, \alpha, l, r, \beta, v\delta y)$ is an edge of G_Ω satifying $\sigma(x\gamma e\delta y) = x\gamma\sigma(e)\delta y$ and $\tau(x\gamma e\delta y) = x\gamma\tau(e)\delta y$, and $(x\gamma e\delta y)^{-1} = x\gamma e^{-1}\delta y$, $\alpha, \beta, \gamma, \delta \in \Omega$. Thus, the free Ω-monoid $MX^*\Omega$ induces a two-sided action on the graph G_Ω. In fact, this action can be extended to paths as follows: if $p = e_1 \circ e_2 \circ \ldots \circ e_n$ and

$x, y \in MX^*\Omega$, then $x\alpha p\beta y = x\alpha_1 e_1 \beta_1 y \circ x\alpha_2 e_2 \beta_2 y \circ \ldots \circ x\alpha_n e_n \beta_n y$ is a path from $\sigma(x\alpha p\beta y) = x\alpha\sigma(p)\beta y$ to $\tau(x\alpha p\beta y) = x\alpha\tau(p)\beta y$, and $(x\alpha p\beta y)^{-1} = x\alpha p^{-1}\beta y$ where $\alpha, \beta, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \Omega$.

Now, let $P(G_\Omega)$ denote the set of all paths in G_Ω, and let

$P^{(2)}(G_\Omega) = \{(p, q) | p, q \in P(G_\Omega), \sigma(p) = \sigma(q), \tau(p) = \tau(q)\}$.

Thus, with $P^{(2)}(G_\Omega)$ we denote the set of all parallel paths in $P(G_\Omega)$. (Two paths p and q are called parallel, which is denoted as $p||q$, if $\sigma(p) = \sigma(q)$ and $\tau(p) = \tau(q)$).

A closed path satisfying $x = \sigma(p) = \tau(p)$ is called a circuit at x and by $P(x)$ we denote the set of all circuits at x. By convention, $P(x)$ always contains the trivial circuit i_x at x.

An equivalence relation $\simeq \subset P^{(2)}(G_\Omega)$ is called a homotopy relation if it satisfies the following conditions:

a) If e_1, e_2 are edges of G_Ω, then $\big(e_1\gamma\sigma(e_2)\big)(\tau(e_1)\gamma e_2) \simeq (\sigma(e_1)\gamma e_2)(e_1\gamma\tau(e_2))$;

b) If $p \simeq q$ $\big(p, q \in P(G_\Omega)\big)$, then $u\alpha p\beta v \simeq u\alpha q\beta v$ for all $u, v \in MX^*\Omega$, $\alpha, \beta \in \Omega$;

c) If $p, q_1, q_2, s \in P(G_\Omega)$ satisfy $\tau(p) = \sigma(q_1) = \sigma(q_2), \tau(q_1) = \tau(q_2) = \sigma(s)$ and $q_1 \simeq q_2$, then $pq_1s \simeq pq_2s$;

d) If $q \in P(G_\Omega)$, then $pp^{-1} \simeq 1_{\sigma(p)}$.

It is seen that the collection of all homotopy relations on $P(G_\Omega)$ is closed under arbitrary intersection, and so $P^{(2)}(G_\Omega)$ itself is a homotopy relation. Thus, if $B \subseteq P^{(2)}(G_\Omega(X; R))$ then there is a unique smallest homotopy relation $\simeq_B$ on $P(G_\Omega(X; R))$ that contains B. This homotopy relation will be called the homotopy relation generated by B.

Definition 9.1. Let $G_\Omega = G_\Omega(X; R)$.

a) D is the following set of pairs: $D =$
$\{(u_1, \alpha_1, l_1, r_1, \beta_1, v_1\gamma u_2\alpha_2 l_2\beta_2 v_2) \circ$
$(u_1\alpha_1 r_1\beta_1 v_1\gamma u_2, \alpha_2\ l_2, r_2, \beta_2, v_2), (u_1\alpha_1 l_1\beta_1 v_1\gamma u_2, \alpha_2, l_2, r_2, \beta_2, v_2) \circ$

$(u_1, \alpha_1, l_1, r_1, \beta_1, v_1\gamma u_2\alpha_2 r_2\beta_2 v_2))|(l_1, r_1), (l_2, r_2) \in R \cup R^{-1},), u_i, v_i \in MX^*\Omega, \alpha_i, \beta_i, \gamma \in \Omega, i = 1,2\}$

b) $I = \{(e \circ e^{-1}, (w))| e\ is\ an\ edge\ of\ G_\Omega\ with\ \sigma(e) = w, w \in MX^*\Omega\}$.

D is called the set of disjoint derivations, while I is the set of inverse derivations. Notice that D and I are subsets of $P^{(2)}(G_\Omega)$.

Theorem 9.2. Let $(X; R)$ be a Ω-monoid presentation, and let G_Ω denote the associated graph, let $B \subseteq P^{(2)}(G_\Omega)$, and let $\approx \subseteq P^{(2)}(G_\Omega)$ be defined as follows:

$\approx = \{(p \circ u\alpha q_1\beta v \circ r, p \circ u\alpha q_2\beta v \circ r)| u, v \in MX^*\Omega, \alpha, \beta \in \Omega, p, r \in P(G_\Omega)\ and\ (q_1, q_2) \in D \cup I \cup B\ such\ that\ \tau(p) = u\alpha\sigma(q_1)\beta v\ and\ \sigma(p) = u\alpha\tau(q_1)\beta v\}$

Then the homotopy relation $\simeq_B$ generated by B is the smallest equivalence relation on $P(G_\Omega)$ that contains the relation $\approx$.

Proof: In the same way as in Theorem 3.4. in [1].

Definition 9.3. Let $(X; R)$ be a Ω-monoid presentation, where Ω is a finite set of binary operations, and let G_Ω denote the associated graph. We say that $(X; R)$ has finite derivation type (FDT) if there is a finite subset $B \subseteq P^{(2)}(G_\Omega)$ which generates $P^{(2)}(G_\Omega)$ as a homotopy relation, i.e., $P^{(2)}(G_\Omega)$ is the only homotopy relation on $P(G_\Omega)$ that contains the set B.

Remark 9.3.1 Another definition for FDT:

For a subset C of || (i.e. of $P^{(2)}(G_\Omega)$) , the homotopy relation $\sim_C$ generated by C is the smallest (with respect to inclusion) homotopy relation containing C. The homotopy relation generated by the empty set $\emptyset$ is denoted by $\sim_0$. If C coincides with ||, then C is called a homotopy base for G_Ω. The presentation $(X; R)$ is said to have finite derivation type (FDT) if the derivation graph G_Ω of $(X; R)$ admits a finite homotopy base where Ω is a finite set of binary operations. A finitely presented Ω-monoid M is said to have FDT if some finite presentation for M has FDT.

Definition 9.4. Let $(X_1; R_1)$ and $(X_2; R_2)$ be two Ω-monoid presentations, let G_Ω^2 denotes the graph associated to $(X_2; R_2)$, and let $f: MX_1^*\Omega \to MX_2^*\Omega$ be a morphism. We call f a mapping of

Ω-monoid presentations if it satisfies the following condition: For all $(l,r) \in R_1$, there is a path in G_Ω^2 from $f(l)$ to $f(r)$.

In the situation of Definition 9.4., we will give some notational conventions. First, for each $(l,r) \in R_1$, we will choose a path $p_{l,r} \in P(G_\Omega^2)$ from $f(l)$ to $f(r)$. If $(f(l), f(r)) \in R_2 \cup R_2^{-1}$, then we choose the corresponding edge of G_Ω^2. If $f(l) = f(r)$, then we choose the path $(f(l))$ of length 0. Next, by $p_{r,l}$ we will denote the path $p_{l,r}^{-1}$ from $f(r)$ to $f(l)$.

Let G_Ω^1 denote the graph associated to $(X_1; R_1)$. Based on the morphism $f: MX_1^*\Omega \to MX_2^*\Omega$ and the choice of paths $p_{l,r}$ we define a mapping $F: G_\Omega^1 \to G_\Omega^2$ as follows: $F = (f_V, f_E)$, where $f_V: MX_1^*\Omega \to MX_2^*\Omega$ is simply the morphism f, and $f_E(u,\alpha,l,r,\beta,v) = f(u)\alpha p_{l,r}\beta f(v)$ for all $u, v \in MX_1^*\Omega$ and $(l,r) \in R \cup R^{-1}$, $\alpha, \beta \in \Omega$. Then $f_E(u,\alpha,l,r,\beta,v)$ is a path in G_Ω^2 from $f(u)\alpha f(l)\beta f(v) = f_V(u\alpha l\beta v) = f_V\big(\sigma_1(u,\alpha,l,r,\beta,v)\big)$ to $f(u)\alpha f(r)\beta f(v) = f_V(u\alpha r\beta v) = f_V\big(\tau_1(u,\alpha,l,r,\beta,v)\big)$. Thus F is a mapping from G_Ω^1 to G_Ω^2 in the sense of definition we gave above. We will say that the mapping $F: G_\Omega^1 \to G_\Omega^2$ exhibits the mapping f from $(X_1; R_1)$ to $(X_2; R_2)$. To simplify the notation we will write F to denote f_V as well as f_E.

Theorem 9.5. Let $(X_1; R_1)$ and $(X_2; R_2)$ be two Ω-monoid presentations with associated graphs G_Ω^1 and G_Ω^2, respectively, let $F: G_\Omega^1 \to G_\Omega^2$ be a mapping that exhibits a mapping f from $(X_1; R_1)$ to $(X_2; R_2)$, let $B_1 \subseteq P^{(2)}(G_\Omega^1)$ and let $\simeq \subseteq P^{(2)}(G_\Omega^2)$ be a homotopy relation. If $F(p) \simeq F(q)$ holds for all $(p,q) \in B_1$, then $F(p) \simeq F(q)$ for all (p,q) satisfying $p \simeq_{B_1} q$.

Proof: Let $\approx_1$ denote the relation on $P(G_\Omega^1)$ that is defined from B_1 as in Theorem 9.2., and let D_1 and I_1 denote the corresponding sets of pairs of paths as they are defined there. Then, $\simeq_{B_1}$ is the equivalence relation on $P(G_\Omega^1)$ generated by $\approx_1$. Using the facts that $\simeq$ is an equivalence relation on $P^{(2)}(G_\Omega^2)$, and F exhibits a mapping f from $(X_1; R_1)$ to $(X_2; R_2)$, it suffices to prove only that from $(p,q) \in D_1 \cup I_1 \cup B_1$ it follows that $F(p) \simeq F(q)$.

If $(p,q) \in B_1$, then $F(p) \simeq F(q)$ by the hypotheses. If $(p,q) \in D_1$, then there are $(l_1, r_1), (l_2, r_2) \in R_1 \cup R_1^{-1}$ such that $p = \big((u_1,\alpha,l_1,r_1,\beta,v_1\gamma u_2\alpha_2 l_2\beta_2 v_2) \circ (u_1\alpha_1 r_1\beta_1 v_1\gamma u_2,\delta,l_2,r_2,\mu,v_2)\big)$ and

$q = \big((u_1\alpha_1 l_1\beta_1 v_1\gamma u_2, \delta, l_2, r_2, \mu, v_2) \circ (u_1, \alpha, l_1, r_1, \beta, v_1\gamma u_2\alpha_2 r_2\beta_2 v_2)\big)$, where $u_i, v_i \in MX_1^*\Omega, i = 1,2, (l_1, r_1), (l_2, r_2) \in R_1$ $\alpha, \beta, \gamma, \delta, \mu \in \Omega$, $\alpha_i, \beta_i \in \Omega, i = 1,2$.

Since F exhibits a mapping f from $(X_1; R_1)$ to $(X_2; R_2)$, we have:

$$F(p) = f_E(u_1, \alpha, l_1, r_1, \beta, v_1\gamma u_2\alpha_2 l_2\beta_2 v_2) = f_V\big(u_1\big)\alpha p_{l_1,r_1}\beta f_V(v_1\gamma u_2\alpha_2 l_2\beta_2 v_2\big)$$

and

$F(q) = f_E(u_1\alpha_1 l_1\beta_1 v_1\delta u_2, \gamma, l_2, r_2, \mu v_2) = f_V(u_1\alpha_1 l_1\beta_1 v_1\gamma u_2)\delta p_{l_2,r_2}\mu f_V(v_2)$

Using the induction on the combined length of the paths p_{l_1,r_1}, p_{l_2,r_2} it is easily verified that $F(p) \simeq F(q)$ holds. If $(p, q) \in I_1$, then there is an edge e of G_Ω^1 with $\sigma_1(e) = w$ such that $p = (w, e, \tau_1(e), e^{-1}, w)$ and $q = (w)$. But $F(p) = f_E(e) \circ f_E(e^{-1}) = p_e \circ p_e^{-1}$ for some path p_e in G_Ω^2 satisfying $\sigma_2(p_e) = f(w)$, and $F(q) = \big(f(w)\big)$. From the definition of homotopy relation, it follows that $p_e \circ p_e^{-1} \simeq (f(w))$ which means that $F(p) \simeq F(q)$ and this completes the proof of the the theorem.

Corollary 9.5.1. Let $(X_1; R_1)$, $(X_2; R_2)$, G_Ω^1 , G_Ω^2, $F: G_\Omega^1 \to G_\Omega^2$ and $B_1 \subseteq P^{(2)}(G_\Omega^1)$ be as in the statement of Theorem 9.5., and $B_2 = \big\{\big(F(p), F(q)\big)\big| p, q \in B_1\big\}$. Then, for all $p, q \in P(G_\Omega^1)$, $p \simeq_{B_1} q$ implies that $F(p) \simeq_{B_2} F(q)$.

The Ω-monoid M presented by $(X; R)$ has infinitely many different finite presentations. We will show that every other finite presentation of the Ω-monoid M has finite derivation type if $(X; R)$ has FDT, i.e., the property of having finite derivation type does not depend on presentation and is an intrinsic property of the Ω-monoid presented. To show this we need the notion of Ω-Tietze transformation.

Definition 9.6. Let $(X; R)$ be a Ω-monoid presentation. The following four types of transformations of $(X; R)$ are called Ω- elementary Tietze transformations:

I. If $u, v \in MX^*\Omega$ such that $u \leftrightarrow_R^* v$, then the presentation $(X; R \cup (u, v))$ is obtained from $(X; R)$ by adding a defining relation,

II. If $u, v \in R$ such that $u \leftrightarrow_{R_1}^{*} v$, where $R_1 = R - (u, v)$, then the presentation $(X; R_1)$ is obtained by $(X; R)$ by deleting a defining relation.

III. If $u \in MX^{*}\Omega$ and $a \notin X$ is a new symbol, then the presentations $(X \cup \{a\}; R \cup (a, u))$ and $(X \cup \{a\}; R \cup (u, a))$ are obtained from $(X; R)$ by adding a generator.

IV. If $a \in X$, and $u \in M(X - \{a\})^{*}\Omega$ such that $(a, u) \in R$ or $(u, a) \in R$, then the presentation $(X - \{a\}; R_1)$ is obtained by $(X; R)$ by deleting a generator. Here, $R_1 = \left\{ (\varphi_a(l), \varphi_a(r)) \middle| (l, r)) \in R - \{(a, u), (u, a)\} \right\}$ where $\varphi_a : MX^{*}\Omega \rightarrow M(X - \{a\})^{*}\Omega$ is the morphism defined through $\varphi_a(b) = b$ if $b \in X - \{a\}$ and $\varphi_a(b) = u$ if $b = a$.

It is easily verified that the Ω-monoids M_{R_1} and M_{R_2} are isomorphic whenever $(X_1; R_1)$ and $(X_2; R_2)$ are two Ω-monoid presentations such that $(X_1; R_1)$ can be transformed into $(X_2; R_2)$ through a finite sequence of Ω- elementary Tietze transformations. The following is the main result on Ω- Tietze transformations.

Proposition 9.7. Let $(X_1; R_1)$ and $(X_2; R_2)$ be two finite presentations of the same Ω-monoid. Then there exists a finite sequence of Ω- elementary Tietze transformations that transforms the presentation $(X_1; R_1)$ into the presentation $(X_2; R_2)$.

For each Ω -elementary Tietze transformation, we will prove a corresponding technical lemma which in essence expresses the fact that if $(X_1; R_1)$ is a finite presentation that has finite derivation type, and if $(X_2; R_2)$ is obtained from $(X_1; R_1)$ by that type of Ω -elementary Tietze transformation, then $(X_2; R_2)$ has finite derivation type as well. From these lemmata and the Proposition 9.7., we get our first main result.

Theorem 9.8. Let $(X_1; R_1)$ and $(X_2; R_2)$ be two finite presentations of the same Ω-monoid. Then the presentation $(X_1; R_1)$ has finite derivation type if and only if the presentation $(X_2; R_2)$ has finite derivation type.

Thus, as we mentioned above, having the finite derivation type is an invariant property of finitely presented Ω-monoids.

Lemma 9.9. Let $(X;R)$ be a finite Ω-monoid presentation, let $u,v \in MX^*\Omega$ be such that $u \leftrightarrow_R^* v$, and let $(X;R_1) = (X;R \cup (u,v))$. If $(X;R)$ has finite derivation type, then so does $(X;R_1)$.

Proof: First, we assume that $u \neq v$ and $(v,u) \notin R$, since for us a string- rewriting system is always irreflexive and anti-symmetric. Let G_Ω denote the graph $G_\Omega(X;R)$ associated with $(X;R)$, and let G_Ω^1 denote the graph $G_\Omega(X;R_1)$ associated with $(X;R_1)$. If $(X;R)$ has finite derivation type, there exist a finite set $B \subseteq P^{(2)}(G_\Omega)$ such that $\simeq_B = P^{(2)}(G_\Omega)$. We will define a finite set $B_1 \subseteq P^{(2)}(G_\Omega^1)$ such that $\simeq_{B_1} = P^{(2)}(G_\Omega^1)$. It can be easily seen that G_Ω^1 is obtained from G_Ω by adding certain edges. We define a mapping from G_Ω^1 to G_Ω as follows. Since $u \leftrightarrow_R^* v$, there is a path $p_{u,v}$ from u to v in G_Ω. On the subgraph $G_\Omega \subseteq G_\Omega^1$, we define f to be the identity. On the additional edges f is defined as follows: $f\big((x,\alpha,u,v,\beta,y)\big) = x\alpha p_{u,v}\beta y$ and $f\big((x,\alpha,v,u,\beta,y)\big) = x\alpha p_{u,v}^{-1}\beta y$ for all $x,y \in MX^*\Omega$. Then $f: G_\Omega^1 \to G_\Omega$ is a mapping of graphs. Define now the set $B_1 \subseteq P^{(2)}(G_\Omega^1)$ as $B_1 = \{\Big(\big(\lambda_\alpha,\alpha,u,v,\beta,\lambda_\beta\big),p_{u,v}\Big),\Big(\big(\lambda_\alpha,\alpha,v,u,\beta,\lambda_\beta\big),p_{u,v}^{-1}\Big)\} \cup B$. Then B_1 is a finite set of pairs of paths in G_Ω^1.

Claim: For all $(p,q) \in P^{(2)}(G_\Omega^1)$, if $f(p) \simeq_B f(q)$, then $p \simeq_{B_1} q$.

Proof: Using the pairs in $B_1 - B$ and the induction on the number of edges from $G_\Omega^1 - G_\Omega$ that occur in p it can be easily verified that $p \simeq_{B_1} f(p)$ for all paths $p \in P(G_\Omega^1)$.

Let $(p,q) \in P^{(2)}(G_\Omega^1)$. Then $(f(p),f(q)) \in P^{(2)}(G_\Omega)$, and hence, if $f(p) \simeq_B f(q)$, then, since $B \subseteq B_1$, this yields $p \simeq_{B_1} f(p) \simeq_{B_1} f(q) \simeq_{B_1} q$.

Since $\simeq_B = P^{(2)}(G_\Omega)$, we have $f(p) \simeq_B f(q)$ for all $(p,q) \in P^{(2)}(G_\Omega^1)$. So, $B_1 \subseteq P^{(2)}(G_\Omega^1)$, which means that $(X;R_1)$ has finite derivation type. This completes the proof of Lemma 9.9.

Lemma 9.10. Let $(X;R_1)$ be a finite Ω-monoid presentation, and let $(u,v) \in R_1$ be such that $u \leftrightarrow_R^* v$, where $R = R_1 - \{(u,v)\}$. If $(X;R_1)$ has finite derivation type, then so does $(X;R)$.

Proof: Let $G_\Omega = G_\Omega(X; R)$, and let $G_\Omega^1 = G_\Omega(X; R_1)$. Then G_Ω is obtained from G_Ω^1 by deleting all edges of the form $(x, \alpha, u, v, \beta, y)$ and $(x, \alpha, v, u, \beta, y)$, $x, y \in MX^*\Omega$. Thus, G_Ω is a subgraph of G_Ω^1. Since, $u \leftrightarrow_R^* v$, we can choose a path $p_{u,v}$ from u to v in G_Ω. We will define , now, a mapping of graphs $f: G_\Omega^1 \to G_\Omega$ in the same way as in the proof of Lemma 9.9. Let $B_1 \subseteq P^{(2)}(G_\Omega^1)$ be such that $\simeq_{B_1} = P^{(2)}(G_\Omega^1)$ and let $B = \{(f(p), f(q)) | (p, q) \in B_1\}$. Thus, if B_1 is finite, then B is a finite subset of $P^{(2)}(G_\Omega)$.

Claim: For all $(p, q) \in P^{(2)}(G_\Omega)$, $p \simeq_B q$.

Proof: Let $(p, q) \in P^{(2)}(G_\Omega)$. Then $(p, q) \in P^{(2)}(G_\Omega^1)$, and hence, $p \simeq_{B_1} q$. Then, by Corollary 9.5.1. it follows that $f(p) \simeq_B f(q)$. But, since $(p, q) \in P^{(2)}(G_\Gamma)$, we have $f(p) = p$ and $f(q) = q$, i.e., $p \simeq_B q$.

Lemma 9.11. Let $(X; R)$ be a finite Ω-monoid presentation, let $u \in MX^*\Omega$, and let $a \notin X$ be a new letter. If $(X; R)$ has finite derivation type, then so does $(X_1; R_1) = (X \cup \{a\}; R \cup (a, u))$.

Proof: Let $G_\Omega = G_\Omega(X; R)$ be the graph associated with $(X; R)$, and let $B \subseteq P^{(2)}(G_\Omega)$ be a finite set such that $\simeq_B = P^{(2)}(G_\Omega)$. Next, let $G_\Omega^1 = G_\Omega(X_1; R_1)$ be the graph associated with $(X_1; R_1)$. We see that G_Ω is a subgraph of G_Ω^1 and define a morphism $f: MX_1^*\Omega \to MX^*\Omega$ by $f(b) = b$ if $b \in X$ and $f(b) = u$ if $b = a$. Then f is a mapping of Ω-monoid presentations, according to the corresponding definition. For each $(l, r) \in R$, we choose the path $(\lambda_\alpha, \alpha, l, r, \beta, \lambda_\beta)$ of length 1 from l to r, and for $(a, u) \in R_1$, we choose the path (u) of length 0 at u. In this way we obtain a mapping $f: G_\Omega^1 \to G_\Omega$ that exhibits the above mapping of Ω-monoid presentations as described after definition we mentioned above. In fact, f maps G_Ω^1 onto G_Ω, and taking its restriction to G_Ω, f is the identity mapping.

Finally, we take $B_1 = B \subseteq P^{(2)}(G_\Omega) \subseteq P^{(2)}(G_\Omega^1)$. By $\simeq_{B_1}$ we denote the homotopy relation on $P(G_\Omega^1)$ that is generated by B_1. We claim that $\simeq_{B_1} = P^{(2)}(G_\Omega^1)$. To prove this claim we will need to prove a sequence of intermediate claims.

Let $\tilde{G}_\Omega$ denote the subgraph of G_Ω^1 that has the same vertices as G_Ω^1, but that contains only those edges $(x, \alpha, l, r, \beta, y)$ of G_Ω^1 for which $(l, r) = (a, u)$ or $(l, r) = (u, a)$. By $P_+(\tilde{G}_\Omega)$ we denote the sets of those paths in $\tilde{G}_\Omega$ that only contain edges of the form $(x, \alpha, a, u, \beta, y)$ where $x, y \in MX_1^*\Omega$, and by $P_-(\tilde{G}_\Omega)$ we denote the set those paths in $\tilde{G}_\Omega$ that contain only edges of the form $(x, \alpha, u, a, \beta, y)$ where $x, y \in MX_1^*\Omega$. Let $\simeq$ denote an arbitrary homotopy relation on $P(G_\Omega^1)$. Then

Claim 1. Let $w \in MX_1^*\Omega$. Then there is a path $p_w \in P_+(\tilde{G}_\Omega)$ from w to $f(w)$, and any two such paths are homotopic mod $\simeq$.

Proof: If $w = v_0\alpha_1 a\beta_1 v_1\alpha_2 a\beta_2 v_2 \dots v_{m-1}\alpha_m a\beta_m v_m$, $v_i \in MX^*\Omega, \alpha_i, \beta_i \in \Omega, i = 0,1, \dots, m$, $a \notin X$ then

$$p_w = (w, (v_0, \alpha_1 a\beta_1, \alpha_1 u\beta_1, v_1\alpha_2 a\beta_2 v_2 \dots v_{m-1}\alpha_m a\beta_m v_m),$$

$$v_0\alpha_1 u\beta_1 v_1\alpha_2 a\beta_2 v_2 \dots v_{m-1}\alpha_m a\beta_m,$$

$$\dots, (v_0\alpha_1 u\beta_1 v_1\alpha_2 a\beta_2 v_2 \dots v_{m-1}, \alpha_1 a\beta_1, \alpha_1 u\beta_1, v_m), f(w))$$

is a path from $P_+(\tilde{G}_\Omega)$ and $\tau(p_w) = f(w)$. If $p' \in P_+(\tilde{G}_\Omega)$ is another path from w to $f(w)$, then p_w and p' are different only in the order in which the occurrences of the letter a are replaced by the string u, and, in this case it is easily verified that $p' \simeq p_w$ holds.

Claim 2. Let $p \in P(\tilde{G}_\Omega)$. Then there exist paths $p_+ \in P_+(\tilde{G}_\Omega)$ and $p_- \in P_-(\tilde{G}_\Omega)$ such that $\sigma(p) = \sigma(p_+)$, $\tau(p_+) = \sigma(p_-)$, $\tau(p_-) = \tau(p)$ and $p \simeq p_+ \circ p_-$.

Proof: Let $p = e_1 \circ e_2 \circ \dots \circ e_m$, where $e_1, \dots, e_m$ are edges of $\tilde{G}_\Omega$. If p itself has not the required form, then there is an index $i < m$ such that $e_i = (x_i, \alpha, u, a, \beta, y_i)$ and $e_{i+1} = (x_{i+1}, \gamma, a, u, \delta, y_{i+1})$. If $x_i = x_{i+1}$, then the edges e_i and e_{i+1} are inverse to each other, and hence, $p \simeq e_1 \circ e_2 \circ \dots e_{i-1} \circ e_{i+2} \circ \dots \circ e_m$. If $x_i \neq x_{i+1}$, then from the relation $x_i\alpha a\beta y_i = x_{i+1}\gamma a\delta y_{i+1}$ it follows that these edges have as necessary consequence the disjoint applications of relations. In fact, if $x_i = x_{i+1}\mu a\vartheta z_{i+1}$ and $y_{i+1} = z_{i+1}\nu a\lambda y_i$ then

$$e_i \circ e_{i+1}$$

$$= (x_{i+1}\mu a\vartheta z_{i+1}\nu u\lambda y_i, x_{i+1}\mu a\vartheta z_{i+1}\nu a\lambda y_i, x_{i+1}\mu u\vartheta z_{i+1}\nu a\lambda y_i)$$

$$= f_i \circ f_{i+1}$$

So, $p \simeq e_1 \circ e_2 \circ \ldots e_{i-1} \circ f_i \circ f_{i+1} \circ e_{i+2} \circ \ldots \circ e_m$.

Repeating this procedure we get that p is transformed into a path of the required form.

Claim 3. Let $(x, \alpha, l, r, \beta, y)$ be an edge of G_Ω^1 such that $(l, r) \in R \cup R^{-1}$, let $p_+ \in P_+(\tilde{G}_\Omega)$ be a path from $x\gamma l\delta y$ to $f(x\gamma l\delta y)$ and let $p_- \in P_-(\tilde{G}_\Omega)$ be a path from $f(x\gamma r\delta y)$ to $x\gamma r\delta y$. Then

$$(x\gamma l\delta y, (x, \alpha, l, r, \beta, y), x\gamma r\delta y) \simeq p_+ \circ \Big(f(x\gamma l\delta y), \big(f(x), \alpha, l, r, \beta, f(y)\big), f(x\gamma r\delta y)\Big) \circ p_-,$$

where $(l, r) \in R \cup R^{-1}$.

Proof: If $x, y \in MX^*\Omega$, then $f(x\gamma l\delta y) = x\gamma l\delta y$ and $f(x\gamma r\delta y) = x\gamma r\delta y$ and in this case, our problem is resolved and so we have nothing to prove. Assume that $x\zeta y$ contains occurrences of the letter a. Thus we will have $x = x_0\alpha_1 a\beta_1 x_1\alpha_2 a \ldots x_{m-1}\alpha_m a\beta_m x_m$ and $y = y_0\mu_1 a\vartheta_1 y_1 \ldots y_{n-1}\mu_n a\vartheta_n y_n$. By Claim 1, there is a path $p_+ \in P_+(\tilde{G}_\Omega)$ from $x\gamma l\delta y = x_0\alpha_1 a\beta_1 x_1\alpha_2 a \ldots x_{m-1}\alpha_m a\beta_m x_m\gamma l\delta y_0\mu_1 a\vartheta_1 y_1 \ldots y_{n-1}\mu_n a\vartheta_n y_n$ to

$f(x\gamma l\delta y) = x_0\alpha_1 u\beta_1 x_1\alpha_2 u \ldots x_{m-1}\alpha_m u\beta_m x_m\gamma l\delta y_0\mu_1 u\vartheta_1 y_1 \ldots y_{n-1}\mu_n u\vartheta_n y_n$ and

$x\gamma r\delta y = x_0\alpha_1 a\beta_1 x_1\alpha_2 a \ldots x_{m-1}\alpha_m a\beta_m x_m\gamma r\delta y_0\mu_1 a\vartheta_1 y_1 \ldots y_{n-1}\mu_n a\vartheta_n y_n$ to

$f(x\gamma r\delta y) = x_0\alpha_1 u\beta_1 x_1\alpha_2 u \ldots x_{m-1}\alpha_m u\beta_m x_m\gamma r\delta y_0\mu_1 u\vartheta_1 y_1 \ldots y_{n-1}\mu_n u\vartheta_n y_n$.

Next, the path $p_+ \circ p_+^{-1}$ is homotopic to the empty path $(x\gamma l\delta y)$, and so $(x\gamma l\delta y, (x, \alpha, l, r, \beta, y), x\gamma r\delta y) \simeq p_+ \circ p_+^{-1} \circ (x, \alpha, l, r, \beta, y)$. Here, we describe paths or parts of paths by displaying the edges used, for simplicity. Since the relations used in the path p_+^{-1} and the relation used on the edge $(x, \alpha, l, r, \beta, y)$ are disjoint, the repeatedly application of the Definition of homotopy gives a path of the form $p_+ \circ (f(x), \alpha, l, r, \beta, f(y)) \circ p_-$ from $x\gamma l\delta y$ to $x\gamma r\delta y$ where $p_- \in P_-(\tilde{G}_\Omega)$.

Claim 4. Let $p \in P(G_\Gamma^1)$. Then there exist paths $p_+ \in P_+\big(\tilde{G}_\Omega\big)$, $q \in P(G_\Omega)$ and $p_- \in P_-\big(\tilde{G}_\Omega\big)$such that $\sigma(p_+) = \sigma(p), \tau(p_+) = f\big(\sigma(p)\big), \sigma(q) = f\big(\sigma(p)\big), \tau(q) = f\big(\tau(p)\big), \sigma(p_-) = f\big(\tau(p)\big)$,

$\tau(p_-) = \tau(p)$, and $p \simeq p_+ \circ q \circ p_-$.

Proof: Let $p \in P(G_\Omega^1)$ be a path from g to h. By Claim 3 we can replace each edge of the form $(x, \alpha, l, r, \beta, y), (l, r) \in R \cup R^{-1}$, by a path from $x\gamma l\delta y$ to $f(x\gamma l\delta y)$, then the edge $(f(x), \alpha, l, r, \beta, f(y))$, and next a path from $f(x\gamma r\delta y)$ to $x\gamma r\delta y$. Thus, we can assume that whenever $(x, \alpha, l, r, \beta, y)$ is an edge of p such that $(l, r) \in R \cup R^{-1}$, then $x, y \in MX^*\Omega$. We can now factor p presenting it in the form $p = p_0 \circ q_1 \circ p_1 \circ \ldots \circ q_n \circ p_n$, where $p_0, p_1, \ldots, p_n \in P(\tilde{G}_\Omega)$ and $q_1, q_2, \ldots, q_n \in P(G_\Omega)$. If $n = 0$, then $f(g) = f(h)$. By Claim 2, $p \simeq p_+ \circ p_-$ for some paths $p_+ \in P_+(\tilde{G}_\Omega)$ and $p_- \in P_-(\tilde{G}_\Omega)$. If $\tau(p_+) \notin MX^*\Omega$, then applying Claim 1 there is a path $p'_+ \in P_+(\tilde{G}_\Omega)$ such that p'_+ leads from $\tau(p_+)$ to $f(\tau(p_+)) = f(g)$. Hence, $p \simeq p_+ \circ p'_+ \circ (p'_+)^{-1} \circ p_-, p_+ \circ p'_+ \in P_+(\tilde{G}_\Omega)$ and $(p'_+)^{-1} \circ p_- \in P_-(\tilde{G}_\Omega)$ satisfying the required properties. So let $n > 0$. But $q_1 \in P(G_\Omega)$, implies that $\sigma(q_1) = \tau(p_0) \in MX^*\Omega$. Thus, $\sigma(p_0) = \sigma(p)$ and $\tau(p_0) = f(\sigma(p))$. Applying Claim 2 we can replace p_0 by a path $p_+ \in P_+(\tilde{G}_\Omega)$ from $\sigma(p)$ to $f(\sigma(p))$. In the same way, we can replace p_n by a path $p_- \in P_-(\tilde{G}_\Omega)$ from $f(\tau(p))$ to $\tau(p)$. Finally, let $i \in \{1,2,\ldots,n-1\}$. Then $p_i \in P(\tilde{G}_\Omega)$ is a path from $\tau(q_i) \in MX^*\Omega$ to $\sigma(q_{i+1})) \in MX^*\Omega$. Since $p_i \in P(\tilde{G}_\Omega)$, we have $\tau(q_i)= \sigma(p_i) = f(\sigma(p_i)) = f(\tau(p_i)) = \sigma(q_{i+1})$ and so, by Claim 2, $p_i \simeq (\sigma(p_i))$. Hence, $p = p_0 \circ q_1 \circ p_1 \circ \ldots \circ q_n \circ p_n \simeq p_+ \circ q_1 \circ q_2 \circ \ldots \circ q_n \circ q_-$. Choosing $q = q_1 \circ q_2 \circ \ldots \circ q_n$ we obtain the required result.

Claim 5. $\simeq_{B_1} = P^{(2)}(G_\Omega^1)$.

Proof: Let $(p, q) \in P^{(2)}(G_\Omega^1)$. Applying Claim 4 we get $p \simeq_{B_1} p_+ \circ p_1 \circ p_-$ and $q \simeq_{B_1} q_+ \circ q_1 \circ q_-$, where $p_+, q_+ \in P_+(\tilde{G}_\Omega), p_-, q_- \in P_-(\tilde{G}_\Omega)$ and $p_1, q_1 \in P(G_\Omega)$ are such that $\sigma(p_1) = f(\sigma(p)), \tau(p_1) = f(\tau(p)), \sigma(q_1) = f(\sigma(q))$ and $\tau(q_1) = f(\tau(q))$. Since $\sigma(p) = \sigma(q)$, we have $\sigma(p_1) = f(\sigma(p)) = f(\sigma(q)) = \sigma(q_1)$, and since $\tau(p) = \tau(q)$, we have $\tau(p_1) = f(\tau(p)) = f(\tau(q)) = \tau(q_1)$, which implies that $(p_1, q_1) \in P^{(2)}(G_\Omega)$. But we choose $B = B_1$ and so $p_1 \simeq_{B_1} q_1$. By Claim 1 $p_+ \simeq_{B_1} q_+$, since $\sigma(p_+) = \sigma(p) = \sigma(q) = \sigma(q_+)$ and $\tau(p_+) = f(\sigma(p)) = f(\sigma(q)) = \tau(q_+)$. In the same way, $p_- \simeq_{B_1} q_-$. Thus, $p \simeq_{B_1} p_+ \circ p_1 \circ p_- \simeq q_+ \circ q_1 \circ q_- \simeq_{B_1} q$.

We conclude that if $(X; R)$ has finite derivation type, then so does $(X_1; R_1)$ and the proof of Lemma 9.11. is completed. We obtain the analogous result if $(X_1; R_1) = (X \cup \{a\}; R \cup (u, a))$.

Lemma 9.12. Let $(X_1; R_1)$ be a finite Ω-monoid presentation, and let $(X; R)$ be obtained from $(X_1; R_1)$ by an Ω- elementary Tietze transformation of type IV. If $(X_1; R_1)$ has finite derivation type, so does $(X; R)$.

Proof: Let $a \in X_1$, let $X = X_1 - \{a\}$, and let $u \in MX^*\Omega$ such that $(a, u) \in R_1$ and $R = \{(\varphi_a(l), \varphi_a(r)) | (l, r) \in R_1 - (a, u)\}$, where $\varphi_a: MX_1^*\Omega \rightarrow MX^*\Omega$ is defined as $\varphi_a(b) = b$ if $b \in X$, $\varphi_a(b) = u$ if $b = a$. Using Ω- Tietze transformations of type I and II the presentation $(X_1; R_1)$ can be transformed into the presentation $(X_1; R \cup (a, u))$. So, if $(X_1; R_1)$ has finite derivation type then, by the above lemmata, this presentation has FDT, as well. Thus, we may assume, without loss of generality, that for all $(l, r) \in R_1 - \{(a, u)\}$, neither l nor r contains an occurrence of the letter a, i.e., $R = R_1 - \{(a, u)\}$. Let $G_\Omega = G_\Omega(X; R)$ and $G_\Omega^1 = G_\Omega(X_1; R_1)$ denote the graphs associated to $(X; R)$ and $(X_1; R_1)$, respectively. Further, let $B_1 \subseteq P^{(2)}(G_\Omega^1)$ be such that $\simeq_{B_1} = P^{(2)}(G_\Omega^1)$. Proceeding as in the proof of Lemma 9.11. we obtain a mapping $f: G_\Omega^1 \rightarrow G_\Omega$ that exhibits the mapping φ_a of Ω-monoid presentations. We now choose $B = \{(f(p), f(q)) | (p, q) \in B_1\} \subseteq P^{(2)}(G_\Omega)$. So, it is enough to prove the following

Claim: For all $(p, q) \in P^{(2)}(G_\Omega)$, $p \simeq_B q$.

Proof: Let $(p, q) \in P^{(2)}(G_\Omega)$. Then $(p, q) \in P^{(2)}(G_\Omega^1)$, and hence $p \simeq_{B_1} q$. By Corollary 9.5.1. this implies that $f(p) \simeq_B f(q)$. But, since $p, q \in P(G_\Omega)$, we have $f(p) = p, f(q) = q$, i.e., $p \simeq_B q$.

10. Finite presentation of M and FDT.

Let R be an Ω-string rewriting system in X. Recall that with $IRR(R)$ we denote the set of all irreducible strings mod$R$. An Ω-string rewriting system R is called normalized if $range(R) \subseteq IRR(R)$, and if, for each rule $(l \rightarrow r) \in R, l \in IRR(R - \{(l \rightarrow r)\})$. A convergent Ω-string rewriting system that is also normalized is called canonical. For each finite convergent Ω-string rewriting system R, it can be determined a finite canonical Ω-string rewriting system system R_1 such that R and R_1 are equivalent in the sense that R and R_1 are defined on the same alphabet and $\leftrightarrow_R^* = \leftrightarrow_{R_1}^*$. This result is proved in the same way as in [5].

An Ω-monoid presentation $(X; R)$ containing a canonical Ω-system will be called a canonical Ω-presentation.

Let (e_1, e_2) be a critical peak of edges. An orderd pair (p_1, p_2) of paths $p_1, p_2 \in P_+(G_\Omega)$ is called a resolution of (e_1, e_2) if $\sigma(p_1) = \tau(e_1), \sigma(p_2) = \tau(e_2)$ and $\tau(p_1) = \tau(p_2)$ hold.

For each critical pair of edges (e_1, e_2), let (p_1, p_2) denote a fixed resolution.

Theorem 10.1. Let $(X; R)$ be a Ω-canonical presentation, let G_Ω be the graph associated to $(X; R)$, and let $B \subseteq P_+^{(2)}(G_\Omega)$ such that B is the set of pairs of the form $(e_1 \circ p_1, e_2 \circ p_2)$ where (e_1, e_2) is a critical peak of edges and (p_1, p_2) is the chosen resolution of (e_1, e_2).

Then $\simeq_B = P^{(2)}(G_\Omega)$ where $\simeq_B$ denote the homotopy relation on $P(G_\Omega)$ that is generated by B.

Observe that the set B is a subset of $P_+^{(2)}(G_\Omega)$ since $e_1 \circ p_1, e_2 \circ p_2 \in P_+(G_\Omega)$ for all pairs $(e_1 \circ p_1, e_2 \circ p_2) \in B$. Note, also, that B is a finite set if R is finite.

So, we obtain immediately our main result:

Theorem 10.2. Let M be a finitely presented Ω-monoid. If M has a presentation $(X; R)$ involving a finite convergent Ω-string rewriting system R, then M has finite derivation type.

Proof: Since M has a finite presentation $(X; R)$ such that R is convergent, it also has a finite canonical presentation $(X_1; R_1)$. The notion of equivalence of string rewriting systems has to do, as we know, with the congruence on the free monoid generated by the alphabets of the two systems. The alphabets must be the same, and the systems are equivalent if and only if they generate the same congruence on the corresponding free monoid. It is easily seen that if $(X; R_1)$ and $(X; R_2)$ are two equivalent string rewriting systems, then the Ω-monoids M_{R_1} and $M_{R_2}$are identical. So, they are isomorphic, as well. Now, M has a finite presentation $(X; R)$ where R is convergent. So, it has a finite canonical presentation $(X_1; R_1)$, as well. The set of critical peaks of R_1 is finite. It follows, from this, that the set B corresponding to $(X_1; R_1)$ is finite. Applying, now, the Theorem 10.1 and Theorem 9.8 we conclude that each finite presentation of M has finite derivation type.

It remains to prove the Theorem 10.1.

First, we will prove the following

Lemma 10.3. Let $w \in MX^*\Omega$ and $z \in IRR(R)$, and let $p_1, p_2 \in P_+(G_\Omega)$ satisfying $\sigma(p_1) = w = \sigma(p_2)$ and $\tau(p_1) = z = \tau(p_2)$. Then $p_1 \simeq_B p_2$.

Proof: Let us apply the Noetherian induction. If w is irreducible, then $w = z$. Since $p_1, p_2 \in P_+(G_\Omega)$, these two paths must be both the corresponding path of length 0, and so $p_1 = p_2$. If w is not irreducible, then both p_1 and p_2 have length larger then 0, since z is irreducible. So, there are edges f_1 and f_2 and paths q_1 and q_2, all from $P_+(G_\Omega)$, such that $p_i = f_i \circ q_i, i = 1,2$. Let $w_i = \tau(f_i) = \sigma(q_i), i = 1,2$.

Claim: There exist a word $w' \in MX^*\Omega$ and paths $g_1, g_2 \in P_+(G_\Omega)$ such that $\sigma(g_i) = w_i, \tau(g_i) = w', i = 1,2$ and $f_1 \circ g_1 \simeq_B f_2 \circ g_2$.

Proof: If $f_1 = f_2$, then $w_1 = w_2$ and we can take g_1 and g_2 to be the corresponding paths of length 0. If $f_1 = (v_0, \alpha, l_1, r_1, \beta, v_1\alpha_2 l_2 \beta_2 v_2)$ and $f_2 = (v_0\alpha_1 l_1 \beta_1 v_1, \gamma, l_2, r_2, \delta, v_2)$ then we choose g_1 to be the path consisting of the single edge $(v_0\alpha_1 r_1 \beta_1 v_1, \gamma, l_2, r_2, \delta, v_2)$ and g_2 to be the path consisting of the single edge $(v_0, \alpha, l_1, r_1, \beta, v_1\alpha_2 r_2 \beta_2 v_2)$. Then from the definition of homotopy relation, it follows immediately that $f_1 \circ g_1 \simeq_B f_2 \circ g_2$. If, now, there are words $x, y \in MX^*\Omega$ and a critical peak of edges (e_1, e_2) such that $f_i = x\alpha_i e_i \beta_i y, i = 1,2$, then we choose $g_i = x\alpha_i q_i' \beta_i y, i = 1,2$ where (q_1', q_2') is the chosen resolution of (e_1, e_2). But, the fact that $(e_1 \circ q_1', e_2 \circ q_2') \in B$, implies that $f_1 \circ g_1 = x\alpha_1 e_1 \beta_1 y \circ x\alpha_1 q_1' \beta_1 y = x\alpha_1(e_1 \circ q_1')\beta_1 y \simeq_B x\alpha_2(e_2 \circ q_2')\beta_2 y = x\alpha_2 e_2 \beta_2 y \circ x\alpha_2 q_2' \beta_2 y = f_2 \circ g_2$. So, the result of the claim is true.

The result of the Lemma 10.3. follows, now, immediately from the facts that R is canonical and by the induction hypotheses.

Lemma 10.4. Let $p \in P(G_\Omega)$ be a path from w_1 to w_2, let $z_1, z_2 \in IRR(R)$ and let $p_1, p_2 \in P_+(G_\Omega)$ be such that $\sigma(p_i) = w_i$ and $\tau(p_i) = z_i, i = 1,2$, and $p \simeq_B p_1 \circ p_2^{-1}$.

Proof: Notice, first, that $\sigma(p) = w_1$ and $\tau(p) = w_2$ imply $w_1 \leftrightarrow_R^* w_2$. Hence, since R is canonical, we have $z_1 = z_2$. So, it remains to see if $p \simeq_B p_1 \circ p_2^{-1}$. We use the induction on the length n of the path p. If $n = 0$, then $w_1 = w_2$, and $p_1 \simeq_B p_2$ by Lemma 10.3., and from this it follows that $p_1 \circ p_2^{-1} \simeq_B (w_1) = p$ (by the definition of homotopy). If $n > 0$, then there exist $w \in MX^*\Omega$, a path $q \in P(G_\Omega)$ from w_1 to w of length $n - 1$, and an edge f of G_Ω from w to w_2 such that $p = q \circ f$. Let q_2 be a path from $P_+(G_\Omega)$ that leads from w to $z_1 = z_2$. By the induction hypothesis we have $q \simeq_B p_1 \circ q_2^{-1}$. If f is an edge from $P_+(G_\Omega)$, then $q_2, f \circ p_2 \in P_+(G_\Omega)$ both lead from w to z_1. Thus, by Lemma 10.3., $q_2 \simeq_B f \circ p_2$. This implies that $p_1 \circ p_2^{-1} \simeq_B p_1 \circ p_2^{-1} \circ f^{-1} \circ f \simeq_B p_1 \circ q_2^{-1} \circ f \simeq_B q \circ f = p$. If f is an edge from $P_-(G_\Omega)$, then $f^{-1} \circ q_2, p_2 \in P_+(G_\Omega)$ both lead from w_2 to z_1. By Lemma 10.3., it follows that $f^{-1} \circ q_2 \simeq_B p_2$, and so, $p_1 \circ p_2^{-1} \simeq_B p_1 \circ q_2^{-1} \circ f \simeq_B q \circ f = p$

Now, the result of Theorem 10.1. follows immediately.

Proof of Theorem 10.1.: Indeed, let $(p, q) \in P^{(2)}(G_\Omega)$ and let $w_1 = \sigma(p)$ and $w_2 = \tau(p)$. Next, let $r_1, r_2 \in P_+(G_\Omega)$ such that $\sigma(r_i) = w_i$ and $\tau(r_i) \in IRR(R)$, $i = 1,2$. Since R is canonical, we note that $\tau(r_1) = \tau(r_2)$, and that $p \simeq_B r_1 \circ r_2^{-1} \simeq_B q$, by Lemma 9.4. Hence, $\simeq_B = P^{(2)}(G_\Omega)$.

References:

1. C.C.Squier, F.Otto,Y.Kobayashi, (1994), A finiteness condition for rewriting systems, *Theoretical Computer Science*,131, 271-294.
2. C.Squier, (1987), Word problems and a homological finiteness condition for monoids, *Journal of Pure and Applied Algebra* 49, 201-217.
3. C.Squier, F.Otto, (1987), The word problem for finitely presented monoids and finite canonical rewriting systems.
4. Y.Lafont, (2006), Algebra and Geometry of Rewriting.
5. R.V.Book, F.Otto, (1993), *String-Rewriting Systems*, Springer-Verlag,New York.
6. P.A. Grillet, (2007), *Abstract Algebra*, 2nd edition, Springer.

7. E.Pasku, (2006), *Finiteness Conditions for Monoids and Small Categories*, PhD Thesis, 18-23.