Fixed Point Theorems in Multiplicative Soft Metric Spaces

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Abstract: Multiplicative metric was introduced in [A.E. Bashirov, E.M. Kurpınar and A. Ozyapıcı, Multiplicative calculus and its applications, J. Math.Analy. App., 337(2008) 36-48]. On the other hand soft set theory was introduced in [D. Molodtsov, Soft set-theory-first results, Comput. Math. Appl.37(1999) 19-31]. Soft metric spaces have been investigated, see for example [Sujoy Das and S. K. Samanta, Soft metric, Annals of Fuzzy Mathematics and Informatics, 6(1) (2013) 77-94]. In the present paper we introduce a concept of soft multiplicative metric spaces, and prove some fixed point theorems of contractive mappings on soft multiplicative metric spaces.

AMS Subject Classification: 47H10, 54E50

I. Basic Notions and Notation

Definition 1 [D. A. Molodtsov, Soft set theory first result, Comput. Math. Appl., 37 (1999), 19 - 31. http://dx.doi.org/10.1016/s0898-1221(99)00056-5]: Let X be an initial universe set, and let E be a set of parameters. The pair (F, E) is called a soft set over X, iff $F: E \to P(X)$, where $P(X)$ is the power set of X.

Definition 2 [P. K. Maji, A. R. Roy and R. Biswas "Soft set theory", Comput. Math. Appl., 45 (2003), 555 - 562. http://dx.doi.org/10.1016/s0898-1221(03)00016-6]: The intersection of two soft sets (F, A) and (G, B) over X is the soft set (H, C) , where $C = A \cap B$, and $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ for every $\varepsilon \in C$. We write $(F, A) \tilde{\cap} (G, B) = (H, C)$.

Definition 3 [P. K. Maji, A. R. Roy and R. Biswas "Soft set theory", Comput. Math. Appl., 45 (2003), 555 - 562. http://dx.doi.org/10.1016/s0898-1221(03)00016-6]: The union of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cap B$, and

$$H(\varepsilon) = \begin{cases} F(\varepsilon), if\ \varepsilon \in A \backslash B \\ G(\varepsilon), if\ \varepsilon \in B \backslash A \\ F(\varepsilon) \cup G(\varepsilon), if\ \varepsilon \in A \cup B \end{cases}$$

, for every $\varepsilon \in C$. We write $(F, A) \tilde{\cup} (G, B) = (H, C)$

Definition 4 [P. K. Maji, A. R. Roy and R. Biswas "Soft set theory", Comput. Math. Appl., 45 (2003), 555 - 562. http://dx.doi.org/10.1016/s0898-1221(03)00016-6]: A soft set (F, A) over X is said to be a null soft set denoted by Φ, if for all $\varepsilon \in A$, $F(\varepsilon) = \phi$ (null set).

Definition 5 [P. K. Maji, A. R. Roy and R. Biswas "Soft set theory", Comput. Math. Appl., 45 (2003), 555 - 562. http://dx.doi.org/10.1016/s0898-1221(03)00016-6]: A soft set (F, A) over X is said to be an absolute soft set if for all $\varepsilon \in A$, $F(\varepsilon) = X$.

Definition 6 [P. Majumdar and S. K. Samanta, On soft mappings, Comput. Math. Appl., 60 (2010) 2666 - 2672. http://dx.doi.org/10.1016/j.camwa.2010.09.004] : The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E)\backslash(G, E)$ is defined as $H(e) = F(e)\backslash G(e)$ for all $e \in E$.

Definition 7 [P. Majumdar and S. K. Samanta, On soft mappings, Comput. Math. Appl., 60 (2010) 2666 - 2672. http://dx.doi.org/10.1016/j.camwa.2010.09.004] : The complement of a soft set (F, A) is denoted by $(F, A)^c$, and is defined by $(F, A)^c = (F^c, A)$, where $F^c: A \to P(X)$ is the mapping given by

$F^c(\alpha) = X - F(\alpha)$, for all $\alpha \in A$.

Definition 8 [Sujoy Das and S. K. Samanta, Soft real sets, soft real numbers and their properties, J. Fuzzy Math., 20 (3) (2012), 551 – 576]: Let $\mathbb{R}$ be the set of real numbers and $\mathfrak{B}(\mathbb{R})$ denote the collection of all nonempty bounded subsets of $\mathbb{R}$, and A taken as a set of parameters, then a mapping $F: A \to \mathfrak{B}(\mathbb{R})$ is called a soft real set. It is denoted by (F, A). If specifically, (F, A) is a singleton soft set, then after identifying (F, A) with the corresponding soft element, it will be called a soft real number.

Remark 9: $\tilde{\mathrm{r}}, \tilde{s}, \tilde{t}$ will denote soft real numbers, and $\bar{r}, \bar{s}, \bar{t}$ will denote a particular type of soft real number such that $\bar{r}(\mu) = r$ for all $\mu \in A$ etc.

Definition 10 [Sujoy Das and S. K. Samanta, Soft real sets, soft real numbers and their properties, J. Fuzzy Math., 20 (3) (2012), 551 – 576]: For two soft real numbers $\tilde{\mathrm{r}}, \tilde{s}$

(a) $\tilde{\mathrm{r}} \leq \tilde{s}$ if $\tilde{\mathrm{r}}(\lambda) \leq \tilde{s}(\lambda)$ for all $\lambda \in A$
(b) $\tilde{\mathrm{r}} \geq \tilde{s}$ if $\tilde{\mathrm{r}}(\lambda) \geq \tilde{s}(\lambda)$ for all $\lambda \in A$
(c) $\tilde{\mathrm{r}} < \tilde{s}$ if $\tilde{\mathrm{r}}(\lambda) < \tilde{s}(\lambda)$ for all $\lambda \in A$
(d) $\tilde{\mathrm{r}} > \tilde{s}$ if $\tilde{\mathrm{r}}(\lambda) > \tilde{s}(\lambda)$ for all $\lambda \in A$

Definition 11 [Sujoy Das and S. K. Samanta, Soft metric, Annals of Fuzzy Mathematics and Informatics, 6 (1) (2013), 77 – 94]: A soft set over X is said to be a soft point if there is exactly one $e \in E$, such that $P(e) = \{x\}$ for some $x \in X$ and $P(e') = \phi$ for all $e' \in E\backslash\{e\}$. We will write $\widetilde{x_e}$.

Definition 12 [Sujoy Das and S. K. Samanta, Soft metric, Annals of Fuzzy Mathematics and Informatics, 6 (1) (2013), 77 – 94]: Two soft points $\widetilde{x_e}$ and $\widetilde{y_e}$ are said to be equal if $e = e'$ and $P(e) = P(e')$, that is, $x = y$. Thus, $\widetilde{x_e} \neq \widetilde{y_e}$ iff $x \neq y$ or $e \neq e'$.

Let $\tilde{X}$ be the absolute soft set, that is, $F(e) = X$, for all $e \in E$, where $(F, E) = \tilde{X}$, and $SP(\tilde{X})$ is the collection of soft points $\tilde{X}$, and $R(E)^*$ denote the set of all nonnegative soft real numbers.

Definition 13 [Sujoy Das and S. K. Samanta, Soft metric, Annals of Fuzzy Mathematics and Informatics, 6 (1) (2013), 77 – 94]: A map $\tilde{d}: SP(\tilde{X}) \times SP(\tilde{X}) \to R(E)^*$ is said to be a soft metric on the soft set $\tilde{X}$, if it satisfies

(a) $\tilde{d}(\widetilde{x_{e_1}}, \widetilde{y_{e_2}}) \geq \bar{0}$ for all $\widetilde{x_{e_1}}, \widetilde{y_{e_2}} \tilde{\in} \tilde{X}$
(b) $\tilde{d}(\widetilde{x_{e_1}}, \widetilde{y_{e_2}}) = \bar{0}$ iff $\widetilde{x_{e_1}} = \widetilde{y_{e_2}}$
(c) $\tilde{d}(\widetilde{x_{e_1}}, \widetilde{y_{e_2}}) = \tilde{d}(\widetilde{y_{e_2}}, \widetilde{x_{e_1}})$ for all $\widetilde{x_{e_1}}, \widetilde{y_{e_2}} \tilde{\in} \tilde{X}$
(d) For all $\widetilde{x_{e_1}}, \widetilde{y_{e_2}}, \widetilde{z_{e_3}} \tilde{\in} \tilde{X}$, $\tilde{d}(\widetilde{x_{e_1}}, \widetilde{z_{e_3}}) \tilde{\leqq} \tilde{d}(\widetilde{x_{e_1}}, \widetilde{y_{e_2}}) + \tilde{d}(\widetilde{y_{e_2}}, \widetilde{z_{e_3}})$

We write $(\tilde{X}, \tilde{d}, E)$ to denote the soft metric space

Now we introduce definition of multiplicative soft metric space as follows:

Definition 14: Let $\tilde{X}$ be a nonempty soft set over E, and let $\tilde{s}: \tilde{X} \times \tilde{X} \to R(E)^*$ be a function satisfying the following

(a) $\tilde{s}(\widetilde{x_{e_1}}, \widetilde{y_{e_2}}) > \bar{1}$ for all $\widetilde{x_{e_1}}, \widetilde{y_{e_2}} \tilde{\in} \tilde{X}$ and $\tilde{s}(\widetilde{x_{e_1}}, \widetilde{y_{e_2}}) = \bar{1}$ iff $\widetilde{x_{e_1}} = \widetilde{y_{e_2}}$
(b) $\tilde{s}(\widetilde{x_{e_1}}, \widetilde{y_{e_2}}) = \tilde{s}(\widetilde{y_{e_2}}, \widetilde{x_{e_1}})$ for all $\widetilde{x_{e_1}}, \widetilde{y_{e_2}} \tilde{\in} \tilde{X}$
(c) For all $\widetilde{x_{e_1}}, \widetilde{y_{e_2}}, \widetilde{z_{e_3}} \tilde{\in} \tilde{X}$, $\tilde{s}(\widetilde{x_{e_1}}, \widetilde{z_{e_3}}) \tilde{\leqq} \tilde{s}(\widetilde{x_{e_1}}, \widetilde{y_{e_2}}) \cdot \tilde{s}(\widetilde{y_{e_2}}, \widetilde{z_{e_3}})$

We call $(\tilde{X}, \tilde{s}, E)$ the multiplicative soft metric space

Example 15: Define $\tilde{s}: R(E)^* \times R(E)^* \to [\bar{1}, \bar{\infty})$ by $\tilde{s}(\widetilde{x_{e_1}}, \widetilde{y_{e_2}}) = \bar{a}^{|\bar{x}-\bar{y}|+|\overline{e_1}-\overline{e_2}|}$, where $\widetilde{x_{e_1}}, \widetilde{y_{e_2}} \tilde{\in} R(E)^*$ and $\bar{a} \tilde{>} \bar{1}$, then $\tilde{s}(\widetilde{x_{e_1}}, \widetilde{y_{e_2}})$ is a multiplicative soft metric.

Definition 16: Let $(\tilde{X}, \tilde{s}, E)$ be a multiplicative soft metric space and $\tilde{r}$ be a nonnegative soft real number. For any $\widetilde{a_e} \tilde{\in} \tilde{X}$, by a multiplicative soft open ball with center $\widetilde{a_e}$ and radius $\tilde{r}$, we mean the collection of soft points of $\tilde{X}$ satisfying $\tilde{s}(\widetilde{x_\lambda}, \widetilde{a_e}) \tilde{<} \tilde{r}$. We write $B(\widetilde{a_e}, \tilde{r}) = \{\widetilde{x_\lambda} \tilde{\in} \tilde{X}: \tilde{s}(\widetilde{x_\lambda}, \widetilde{a_e}) \tilde{<} \tilde{r}\}$. The multiplicative soft closed ball is defined as $B[\widetilde{a_e}, \tilde{r}] = \{\widetilde{x_\lambda} \tilde{\in} \tilde{X}: \tilde{s}(\widetilde{x_\lambda}, \widetilde{a_e}) \tilde{\leqq} \tilde{r}\}$.

Definition/Lemma 17: Let $\{\widetilde{x_{\lambda,n}}\}_n$ be a sequence of points in a multiplicative soft metric space $(\tilde{X}, \tilde{s}, E)$. We say $\{\widetilde{x_{\lambda,n}}\}_n$ converges in $(\tilde{X}, \tilde{s}, E)$ if there exist a soft point $\widetilde{y_\mu} \tilde{\in} \tilde{X}$ such that

$$\tilde{s}(\widetilde{x_{\lambda,n}}, \widetilde{y_\mu}) \xrightarrow{multiplicative\ soft\ converges} \bar{1}.$$

Definition 18: Let $(\tilde{X}, \tilde{s}, E)$ be a multiplicative soft metric space and $\{\widetilde{x_{\lambda,n}}\}_n$ be a sequence in $(\tilde{X}, \tilde{s}, E)$. We say $\{\widetilde{x_{\lambda,n}}\}_n$ is a multiplicative soft Cauchy sequence if it holds that for all $\tilde{\varepsilon} \tilde{>} \bar{1}$, choosen arbitrarily, there exist a natural number $N(\tilde{\varepsilon})$ such that $\tilde{s}(\widetilde{x_{\lambda,n}}, \widetilde{x_{\lambda,m}}) \tilde{<} \tilde{\varepsilon}$, for all $m, n \tilde{\geqq} N(\tilde{\varepsilon})$, that is,

$$\tilde{s}(\widetilde{x_{\lambda,n}}, \widetilde{x_{\lambda,m}}) \xrightarrow{multiplicative\ soft\ converges} \bar{1}\ .$$

Definition 19: Let $(\tilde{X}, \tilde{s}, E)$ and $(\tilde{Y}, \tilde{\rho}, E')$ be two multiplicative soft metric spaces. The mapping $(f, \varphi): (\tilde{X}, \tilde{s}, E) \to (\tilde{Y}, \tilde{\rho}, E')$, where $f: X \to Y$ and $\varphi: E \to E'$ is said to be multiplicative soft continuous at the point $\widetilde{x_\lambda} \tilde{\in} \tilde{X}$ iff whenever $\{\widetilde{x_{\lambda,n}}\}_n$ multiplicative soft converges to $\widetilde{x_\lambda}$, $(f, \varphi)(\widetilde{x_{\lambda,n}})$ multiplicative soft converges to $(f, \varphi)(\widetilde{x_\lambda})$.

Definition 20: We say $(\tilde{X}, \tilde{s}, E)$ is multiplicative soft complete, if every multiplicative soft sequence in $(\tilde{X}, \tilde{s}, E)$ that is Cauchy, multiplicative soft converges to a point in $(\tilde{X}, \tilde{s}, E)$

Definition 21: Let $(\tilde{X}, \tilde{s}, E)$ be a multiplicative soft metric space. A function $(f, \varphi): (\tilde{X}, \tilde{s}, E) \to (\tilde{X}, \tilde{s}, E)$ is called a multiplicative soft contraction mapping if there exists $\bar{0} \tilde{\leq} \tilde{\alpha} \tilde{<} \bar{1}$, such that for every, $\widetilde{x_\lambda}, \widetilde{y_\mu} \tilde{\in} \tilde{X}$, we have $\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda}), (f,\varphi)(\widetilde{y_\mu})\right) \tilde{\leq} \tilde{s}(\widetilde{x_\lambda}, \widetilde{y_\mu})^{\tilde{\alpha}}$

II. Main Result

Theorem 22 (Banach Type Contraction Principle): Let $(\tilde{X}, \tilde{s}, E)$ be a multiplicative soft complete metric space. If the mapping $(f, \varphi): (\tilde{X}, \tilde{s}, E) \to (\tilde{X}, \tilde{s}, E)$ is a multiplicative soft contraction mapping on a complete multiplicative soft metric space, then (f, φ) has a unique fixed point.

Proof: Let $\widetilde{x_\lambda^0}$ be any point in $\tilde{X}$. Put $\widetilde{x_{\lambda_{n+1}}^{n+1}} = \left((f,\varphi)(\widetilde{x_{\lambda_n}^n})\right) = \left(f^{n+1}\left(\widetilde{x_\lambda^0}\right)\right)_{\varphi^{n+1}(\lambda)}$

Then we have, $\tilde{s}\left(\widetilde{x_{\lambda_{n+1}}^{n+1}}, \widetilde{x_{\lambda_n}^n}\right) =$

$\tilde{s}\left((f,\varphi)(\widetilde{x_{\lambda_n}^n}), (f,\varphi)\left(\widetilde{x_{\lambda_{n-1}}^{n-1}}\right)\right) \tilde{\leq} \tilde{s}\left(\widetilde{x_{\lambda_n}^n}, \widetilde{x_{\lambda_{n-1}}^{n-1}}\right)^{\alpha} \tilde{\leq} \cdots \tilde{\leq} \tilde{s}\left(\widetilde{x_{\lambda_1}^1}, \widetilde{x_{\lambda_0}^0}\right)^{\alpha^n}$

So for $n > m$, we have that,

$\tilde{s}(\widetilde{x_{\lambda_n}^n}, \widetilde{x_{\lambda_m}^m}) \tilde{\leq} \tilde{s}\left(\widetilde{x_{\lambda_n}^n}, \widetilde{x_{\lambda_{n-1}}^{n-1}}\right) \cdots \tilde{s}\left(\widetilde{x_{\lambda_{m+1}}^{m+1}}, \widetilde{x_{\lambda_m}^m}\right) \tilde{\leq} \tilde{s}\left(\widetilde{x_{\lambda_1}^1}, \widetilde{x_{\lambda_0}^0}\right)^{\alpha^{n-1}+\cdots+\alpha^m} \tilde{\leq} \tilde{s}\left(\widetilde{x_{\lambda_1}^1}, \widetilde{x_{\lambda_0}^0}\right)^{\frac{\alpha^m}{1-\alpha}}$

which implies that $\tilde{s}(\widetilde{x_{\lambda_n}^n}, \widetilde{x_{\lambda_m}^m})$ soft multiplicative converges to $\bar{1}$ as $n, m \to \infty$. So $\{\widetilde{x_{\lambda_n}^n}\}$ is a multiplicative soft Cauchy sequence. Since $\tilde{X}$ is complete, there exists $\widetilde{x_\lambda^*} \tilde{\in} \tilde{X}$ such that $\widetilde{x_{\lambda_n}^n}$ soft multiplicative converges to $\widetilde{x_\lambda^*}$ as $n \to \infty$. Now,

$\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}), \widetilde{x_\lambda^*}\right) \tilde{\leq} \tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}), (f,\varphi)(\widetilde{x_{\lambda_n}^n})\right) . \tilde{s}\left(\widetilde{x_\lambda^*}, (f,\varphi)(\widetilde{x_{\lambda_n}^n})\right) \tilde{\leq} \tilde{s}(\widetilde{x_\lambda^*}, \widetilde{x_{\lambda_n}^n})^{\alpha} . \tilde{s}\left(\widetilde{x_{\lambda_{n+1}}^{n+1}}, \widetilde{x_\lambda^*}\right)$

Going in the limit of the above inequality as $n \to \infty$, we see that $\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}), \widetilde{x_\lambda^*}\right)$ soft multiplicative converges to $\bar{1}$, that is , $\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}), \widetilde{x_\lambda^*}\right) = \bar{1}$. So, $(f,\varphi)(\widetilde{x_\lambda^*}) = \widetilde{x_\lambda^*}$, that is, $\widetilde{x_\lambda^*}$ is a fixed point of (f, φ). Now we show the uniqueness of the fixed point. Suppose $\widetilde{y_\mu^*}$ is another fixed point of (f, φ), then,

$\tilde{s}(\widetilde{x_\lambda^*}, \widetilde{y_\mu^*}) = \tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}), (f,\varphi)(\widetilde{y_\mu^*})\right) \tilde{\leq} \tilde{s}(\widetilde{x_\lambda^*}, \widetilde{y_\mu^*})^{\alpha}$, thus, $\tilde{s}(\widetilde{x_\lambda^*}, \widetilde{y_\mu^*}) = \bar{1}$, that is, $\widetilde{x_\lambda^*} = \widetilde{y_\mu^*}$. So the fixed point is unique. ■

Corollary 23: Let $(\tilde{X}, \tilde{s}, E)$ be a complete multiplicative soft metric space. If $(f, \varphi): (\tilde{X}, \tilde{s}, E) \to (\tilde{X}, \tilde{s}, E)$ satisfies for some positive integer n, $\tilde{s}\left((f^n,\varphi)(\widetilde{x_\lambda}), (f^n,\varphi)(\widetilde{y_\mu})\right) \tilde{\leq} \tilde{s}(\widetilde{x_\lambda}, \widetilde{y_\mu})^{\tilde{\alpha}}$ for all $\widetilde{x_\lambda}, \widetilde{y_\mu} \tilde{\in} \tilde{X}$, where $\bar{0} \tilde{\leq} \tilde{\alpha} \tilde{<} \bar{1}$, then, $(f, \varphi): (\tilde{X}, \tilde{s}, E) \to (\tilde{X}, \tilde{s}, E)$ has a unique fixed point in $(\tilde{X}, \tilde{s}, E)$.

Proof: By previous theorem (f^n, φ) has unique fixed point $\widetilde{x_\lambda^*}$, but

$(f^n,\varphi)\left((f,\varphi)(\widetilde{x_\lambda^*})\right) = (f,\varphi)\left((f^n,\varphi)(\widetilde{x_\lambda^*})\right) = (f,\varphi)(\widetilde{x_\lambda^*})$, so $(f,\varphi)(\widetilde{x_\lambda^*})$ is also a fixed point of (f^n,φ), but $(f,\varphi)(\widetilde{x_\lambda^*}) = \widetilde{x_\lambda^*}$. It follows that the fixed point of (f,φ) is also a fixed point of (f^n,φ), thus the fixed point of (f,φ) is unique ■

Theorem 24: Let $(\tilde{X},\tilde{s},E)$ be a complete soft multiplicative metric space. Suppose that the mapping $(f,\varphi)\colon(\tilde{X},\tilde{s},E)\to(\tilde{X},\tilde{s},E)$ satisfies
$\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda}),(f,\varphi)(\widetilde{y_\mu})\right) \widetilde{\leqq} \left(\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda}),\widetilde{x_\lambda}\right).\tilde{s}\left((f,\varphi)(\widetilde{y_\mu}),\widetilde{y_\mu}\right)\right)^\alpha$ for all $\widetilde{x_\lambda},\widetilde{y_\mu}\,\tilde{\in}\,\tilde{X}$, where $\bar{0}\,\widetilde{\leqq}\,\alpha\,\widetilde{<}\,\frac{\bar{1}}{2}$. Then, (f,φ) has a unique fixed point in $\tilde{X}$, and for any $\widetilde{x_\lambda}\,\tilde{\in}\,\tilde{X}$, the sequence $\{(f^n,\varphi)(\widetilde{x_\lambda})\}$ converges to the fixed point.

Proof: Let $\widetilde{x_\lambda^0}$ be any point in $\tilde{X}$. Put $\widetilde{x_{\lambda_{n+1}}^{n+1}} = \left((f,\varphi)(\widetilde{x_{\lambda_n}^n})\right) = \left(f^{n+1}\left(\widetilde{x_\lambda^0}\right)\right)_{\varphi^{n+1}(\lambda)}$, then,

$$\tilde{s}\left(\widetilde{x_{\lambda_{n+1}}^{n+1}},\widetilde{x_{\lambda_n}^n}\right) =$$
$$\tilde{s}\left((f,\varphi)(\widetilde{x_{\lambda_n}^n}),(f,\varphi)\left(\widetilde{x_{\lambda_{n-1}}^{n-1}}\right)\right) \widetilde{\leqq} \left(\tilde{s}\left((f,\varphi)(\widetilde{x_{\lambda_n}^n}),\widetilde{x_{\lambda_n}^n}\right).\tilde{s}\left((f,\varphi)\left(\widetilde{x_{\lambda_{n-1}}^{n-1}}\right),\widetilde{x_{\lambda_{n-1}}^{n-1}}\right)\right)^\alpha =$$
$$\left(\tilde{s}\left(\widetilde{x_{\lambda_{n+1}}^{n+1}},\widetilde{x_{\lambda_n}^n}\right).\tilde{s}\left(\widetilde{x_{\lambda_n}^n},\widetilde{x_{\lambda_{n-1}}^{n-1}}\right)\right)^\alpha$$

So, $\tilde{s}\left(\widetilde{x_{\lambda_{n+1}}^{n+1}},\widetilde{x_{\lambda_n}^n}\right) \widetilde{\leqq} \tilde{s}\left(\widetilde{x_{\lambda_n}^n},\widetilde{x_{\lambda_{n-1}}^{n-1}}\right)^h$, where $h=\frac{\alpha}{1-\alpha}$. So for $n>m$, we have that,

$$\tilde{s}\left(\widetilde{x_{\lambda_n}^n},\widetilde{x_{\lambda_m}^m}\right) \widetilde{\leqq} \tilde{s}\left(\widetilde{x_{\lambda_n}^n},\widetilde{x_{\lambda_{n-1}}^{n-1}}\right)\cdots\tilde{s}\left(\widetilde{x_{\lambda_{m+1}}^{m+1}},\widetilde{x_{\lambda_m}^m}\right) \widetilde{\leqq} \tilde{s}\left(\widetilde{x_{\lambda_1}^1},\widetilde{x_{\lambda_0}^0}\right)^{h^{n-1}+\cdots+h^m} \widetilde{\leqq} \tilde{s}\left(\widetilde{x_{\lambda_1}^1},\widetilde{x_{\lambda_0}^0}\right)^{\frac{h^m}{1-h}}$$

which implies that $\tilde{s}\left(\widetilde{x_{\lambda_n}^n},\widetilde{x_{\lambda_m}^m}\right)$ soft multiplicative converges to $\bar{1}$ as $n,m\to\infty$. So $\{\widetilde{x_{\lambda_n}^n}\}$ is a multiplicative soft Cauchy sequence. Since $\tilde{X}$ is complete, there exists $\widetilde{x_\lambda^*}\,\tilde{\in}\,\tilde{X}$ such that $\widetilde{x_{\lambda_n}^n}$ soft multiplicative converges to $\widetilde{x_\lambda^*}$ as $n\to\infty$. Now,

$$\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}),\widetilde{x_\lambda^*}\right) \widetilde{\leqq} \tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}),(f,\varphi)(\widetilde{x_{\lambda_n}^n})\right).\tilde{s}\left(\widetilde{x_\lambda^*},(f,\varphi)(\widetilde{x_{\lambda_n}^n})\right)$$
$$\widetilde{\leqq} \left(\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}),\widetilde{x_\lambda^*}\right).\tilde{s}\left((f,\varphi)(\widetilde{x_{\lambda_n}^n}),\widetilde{x_{\lambda_n}^n}\right)\right)^\alpha.\tilde{s}\left(\widetilde{x_{\lambda_{n+1}}^{n+1}},\widetilde{x_\lambda^*}\right)$$
$$= \left(\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}),\widetilde{x_\lambda^*}\right).\tilde{s}\left(\widetilde{x_{\lambda_{n+1}}^{n+1}},\widetilde{x_{\lambda_n}^n}\right)\right)^\alpha.\tilde{s}\left(\widetilde{x_{\lambda_{n+1}}^{n+1}},\widetilde{x_\lambda^*}\right)$$

So it follows that,

$$\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}),\widetilde{x_\lambda^*}\right) \widetilde{\leqq} \tilde{s}\left(\widetilde{x_{\lambda_{n+1}}^{n+1}},\widetilde{x_{\lambda_n}^n}\right)^{\frac{\alpha}{1-\alpha}}.\tilde{s}\left(\widetilde{x_{\lambda_{n+1}}^{n+1}},\widetilde{x_\lambda^*}\right)^{\frac{1}{1-\alpha}}$$

Going in the limit of the above inequality as $n\to\infty$, we see that $\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}),\widetilde{x_\lambda^*}\right)$ soft multiplicative converges to $\bar{1}$, that is , $\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}),\widetilde{x_\lambda^*}\right)=\bar{1}$. So, $(f,\varphi)(\widetilde{x_\lambda^*})=\widetilde{x_\lambda^*}$, that is, $\widetilde{x_\lambda^*}$ is a fixed point of (f,φ). Suppose $\widetilde{y_\mu^*}$ is another fixed point of (f,φ), then,

$$\tilde{s}(\widetilde{x_\lambda^*}, \widetilde{y_\mu^*}) = \tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}), (f,\varphi)(\widetilde{y_\mu^*})\right) \widetilde{\leq} \left(\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda^*}), \widetilde{x_\lambda^*}\right).\tilde{s}\left((f,\varphi)(\widetilde{y_\mu^*}), \widetilde{y_\mu^*}\right)\right)^\alpha$$
$$= \left(\tilde{s}(\widetilde{x_\lambda^*}, \widetilde{x_\lambda^*}).\tilde{s}(\widetilde{y_\mu^*}, \widetilde{y_\mu^*})\right)^\alpha = \bar{1}^\alpha = \bar{1}$$

So, $\widetilde{x_\lambda^*} = \widetilde{y_\mu^*}$, and the fixed point is unique ■

Remark 25: If we replace $\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda}), (f,\varphi)(\widetilde{y_\mu})\right) \widetilde{\leq} \left(\tilde{s}((f,\varphi)(\widetilde{x_\lambda}), \widetilde{x_\lambda}).\tilde{s}\left((f,\varphi)(\widetilde{y_\mu}), \widetilde{y_\mu}\right)\right)^\alpha$ in Theorem 24 with $\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda}), (f,\varphi)(\widetilde{y_\mu})\right) \widetilde{\leq} \left(\tilde{s}\left((f,\varphi)(\widetilde{x_\lambda}), \widetilde{y_\mu}\right).\tilde{s}\left((f,\varphi)(\widetilde{y_\mu}), \widetilde{x_\lambda}\right)\right)^\alpha$, then it still holds.

III. Concluding Remarks

In the present paper we have introduced a notion of multiplicative soft metric spaces, and proved some fixed point theorems in this setting. Saluja et.al [International Journal of Mathematical Analysis Vol. 8, 2014, no. 57, 2809 – 2825] have studied fixed point theorems for weakly contractive maps in generalized soft metric spaces. An interesting problem is to study fixed point theorems for weakly contractive mappings in multiplicative soft metric spaces.

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